ASTRON

Netherlands Institute for Radio Astronomy

Introduction to Radio Astronomy and Interferometry

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Preamble

- **AIM:** This lecture aims to give a general introduction to radio astronomy and interferometry, focusing on the issues that you should consider and the differences with observations with other telescopes.

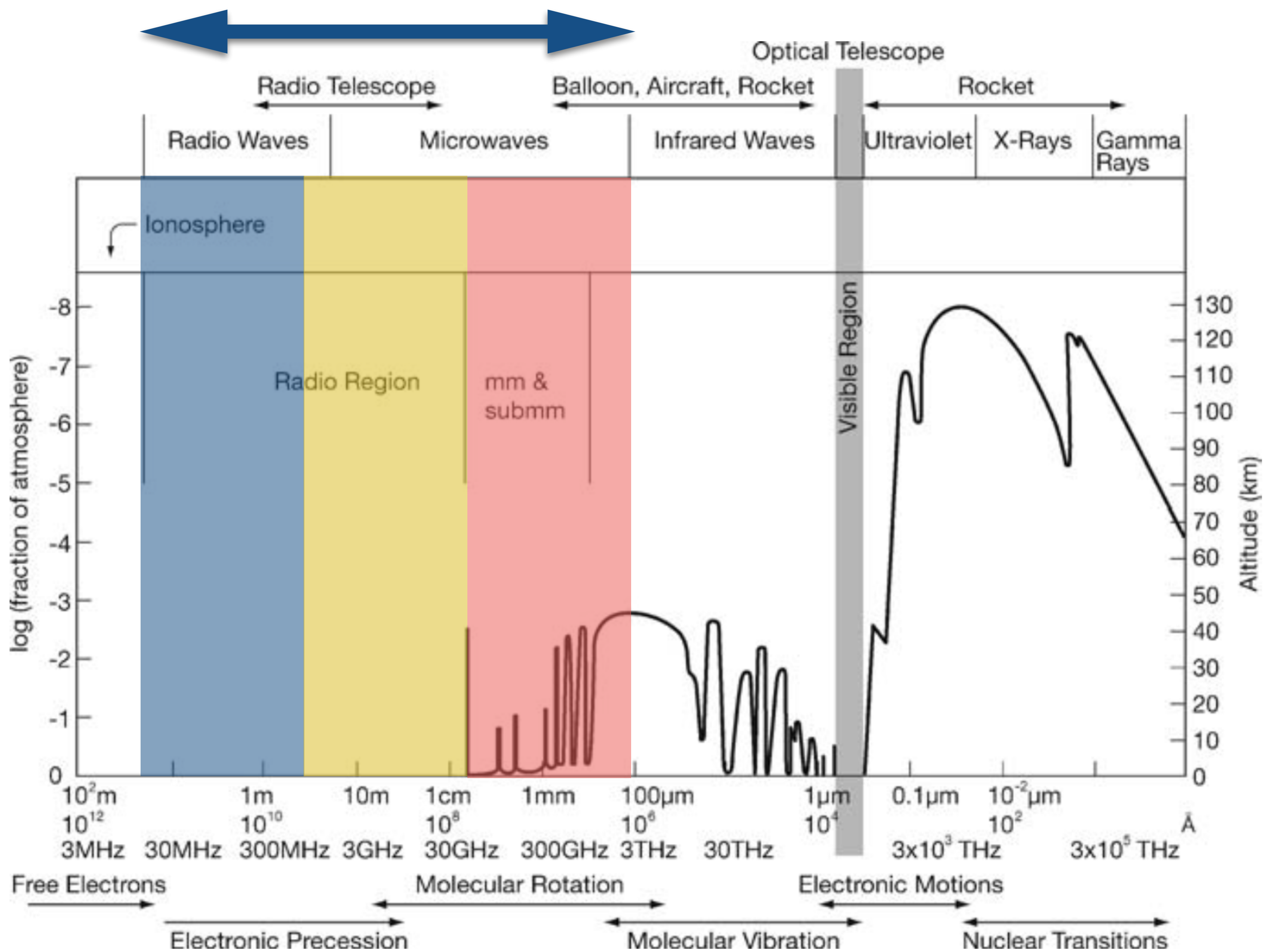
- **OUTLINE:**

  1. The radio sky and historical developments.
  2. The response of a dipole antenna.
  3. The response of a dish antenna.
  4. The response of an interferometer.

1. The radio sky and historical developments

1.1 The radio window

- Radio Astronomy is the study of radiation from celestial sources at frequencies between $\nu \sim$ 10 MHz to 1 THz (10^7 Hz to 10^{12} Hz).



- The observing window is constrained by atmospheric absorption / emission and refraction.
  1) Charged particles in the ionosphere reflects radio waves back into space at < 10 MHz.
  2) Vibrational transitions of molecules have similar energy to infra-red photons and absorb the radiation at > 1 GHz (completely by ~300 GHz).

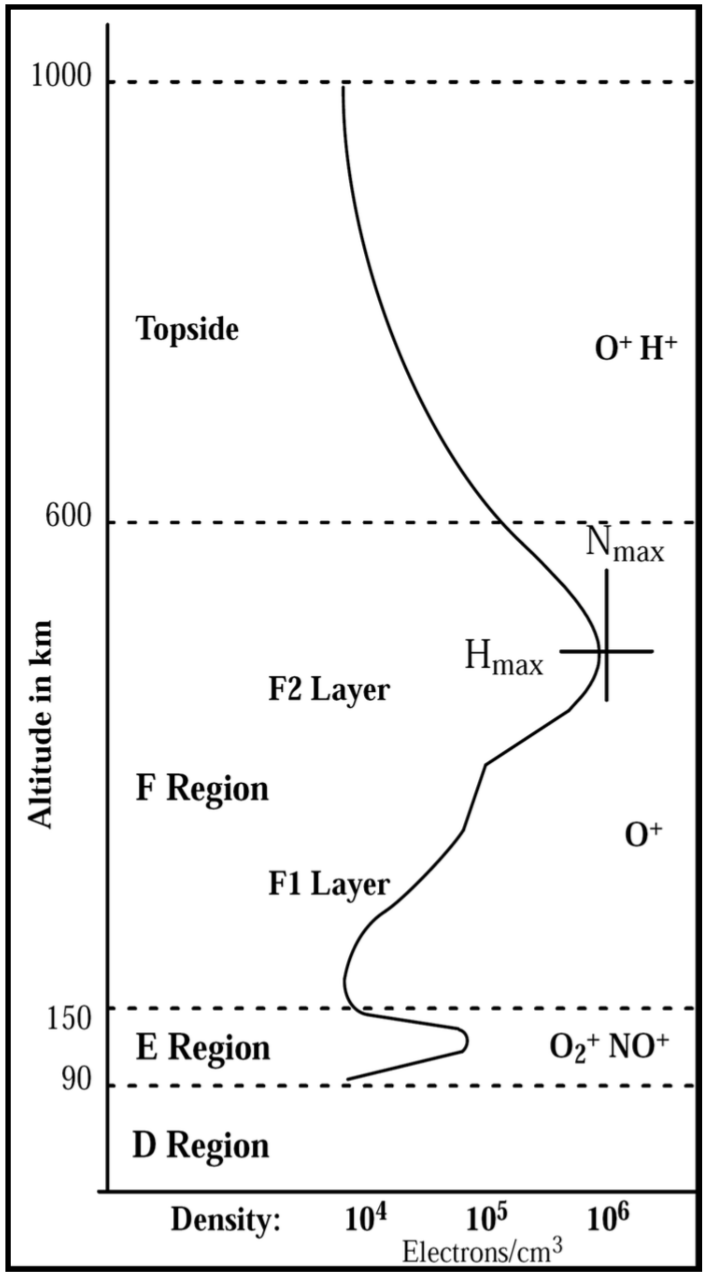
1.2 The low-frequency cut-off (LOFAR)

- The ionosphere consists of a plasma of charged particles (conducting layers).

- The observing conditions are dependent on the electron density, i.e. the solar conditions (space weather), since the ionisation is due to the ultra-violet radiation field from the Sun,

$$O_2 + h\nu \rightarrow O_2^{+*} + e^-$$

$$O_2 + h\nu \rightarrow O^+ + O + e^-$$



1.3 Propagation of radio waves through a (cold) conducting medium (LOFAR, e-MERLIN, EVN)

- A plasma consists of an ionised gas of ions and free electrons that has no net charge. A cold plasma is one where the thermal motions of the electrons is negligible.

- Important for understanding
  1. the reflection and transmission through our atmosphere; and
  2. the dispersion of radio waves at low frequencies.

- As we are dealing with the propagation of radio waves through a conducting medium, we must start with Maxwell's equations.

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Electric field intensity

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Magnetic induction

Current density

where,

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

permeability

permittivity

$$\vec{J} = \sigma \vec{E}$$

Conductivity

- First, consider the curl of the B-field in terms of the E-field, and take the conductivity into account,

$$\nabla \times \vec{B} = \mu_0 \sigma \vec{E} + \mu_0 \epsilon_0 \dot{\vec{E}}$$

- Next we have to take the curl of the E-field and differentiate with respect to time,

$$\nabla \times (\nabla \times \vec{E}) = \frac{d}{dt}(\nabla \times \vec{B}) = \mu_0 \sigma \dot{\vec{E}} + \mu_0 \epsilon_0 \ddot{\vec{E}}$$

$$\nabla^2 \vec{E} = \mu_0 \sigma \dot{\vec{E}} + \mu_0 \epsilon_0 \ddot{\vec{E}}$$

$$\nabla^2 \vec{E} - \mu_0 \sigma \dot{\vec{E}} - \mu_0 \epsilon_0 \ddot{\vec{E}} = 0$$

- This gives the wave equation for the electric field in a conducting material, which we can evaluate by considering a solution given by a harmonic wave of the form,

$$E(r,t) = E_0 e^{-i(wt-kr)}$$

$$\nabla^2 E(r,t) = i^2 k^2 E(r,t)$$

$$\dot{E}(r,t) = E_0 e^{-i(wt-kr)} \cdot -i\omega = -i\omega E(r,t)$$

$$\ddot{E}(r,t) = -i\omega E_0 e^{-i(wt-kr)} \cdot -i\omega = i^2 \omega^2 E(r,t)$$

giving

$$i^2 k^2 \vec{E}(r,t) - \mu_0 \sigma \cdot -i\omega \vec{E}(r,t) - \mu_0 \epsilon_0 \cdot i^2 \omega^2 \vec{E}(r,t) = 0$$

$$-k^2 + i\mu_0 \sigma \omega + \mu_0 \epsilon_0 \omega^2 = 0$$

$$\boxed{k^2 = \frac{\omega^2}{c^2} + i\frac{\sigma\,\omega}{c^2\,\epsilon_0}}$$

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

- Free electrons in the plasma are accelerated by the E-field, with an equation of motion,

$$m_e \dot{v} = -e\,\vec{E}(r,t)$$

with solution,

$$v = -i\frac{e}{m_e \omega}\vec{E}(r,t)$$

- These motions of the charge will result in a current with a density of,

$$\vec{J}(r,t) = -n_e e v = i\frac{n_e e^2}{m_e \omega}\vec{E}(r,t) = \sigma \vec{E}(r,t)$$

where the conductivity is purely imaginary

$$\sigma = i\frac{n_e e^2}{m_e \omega}$$

- Recall our equation of the wave vector

$$k^2 = \frac{\omega^2}{c^2} + i\frac{\sigma\,\omega}{c^2\,\epsilon_0}$$

$$k^2 = \frac{\omega^2}{c^2} + i\frac{\omega}{c^2\epsilon_0} \cdot i\frac{n_e e^2}{m_e \omega}$$

$$= \frac{\omega^2}{c^2} - \frac{n_e e^2}{c^2 \epsilon_0 m_e}$$

$$= \frac{\omega^2}{c^2}\left(1 - \frac{n_e e^2}{\omega^2 \epsilon_0 m_e}\right)$$

$$k^2 = \frac{\omega^2}{c^2}\left(1 - \frac{\omega_p^2}{\omega^2}\right) \quad \text{where} \quad \omega_p = \sqrt{\frac{n_e e^2}{\epsilon_0 m_e}}$$

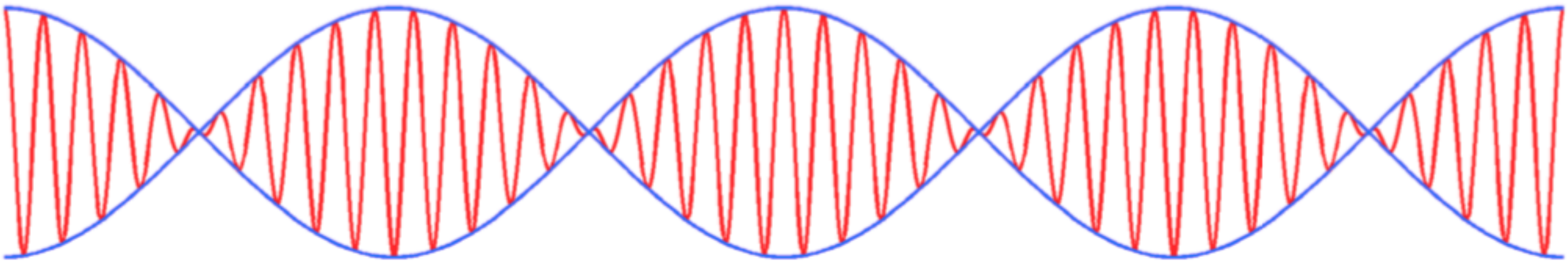
- The plasma frequency defines the natural resonant frequency of a plasma oscillation and is dependent purely on the number density of the free electrons (in free-space).

- **Phase velocity:** the rate that any one frequency component travels through a medium.

$$v_p \equiv \frac{\omega}{k}$$

- **Group velocity:** the rate that the wave envelop travels through a medium.

$$v_g \equiv \frac{d\omega}{dk}$$



- Substituting our equation for the wave vector into the equation for the phase velocity gives,

$$v_p^2 \equiv \frac{\omega^2}{k^2} = \frac{\omega^2}{\frac{\omega^2}{c^2}\left(1 - \frac{\omega_p^2}{\omega^2}\right)}$$

$$v_p = \frac{c}{\sqrt{\left(1 - \frac{\omega_p^2}{\omega^2}\right)}}$$

- Similarly, we can calculate the group velocity as,

$$v_g \equiv \frac{d\omega}{dk} = \frac{1}{dk/d\omega} \qquad v_g = c\sqrt{1 - \frac{\omega_p^2}{\omega^2}}$$

- Both the group and phase velocities are dependent on frequency, but when $\omega < \omega_p$, then the group velocity is < 0 and waves cannot propagate through the plasma.

- From the definition of the refractive index and taking the phase velocity,

$$n = \sqrt{1 - \frac{\omega_p^2}{\omega^2}} \qquad\qquad n = \frac{c}{v}$$

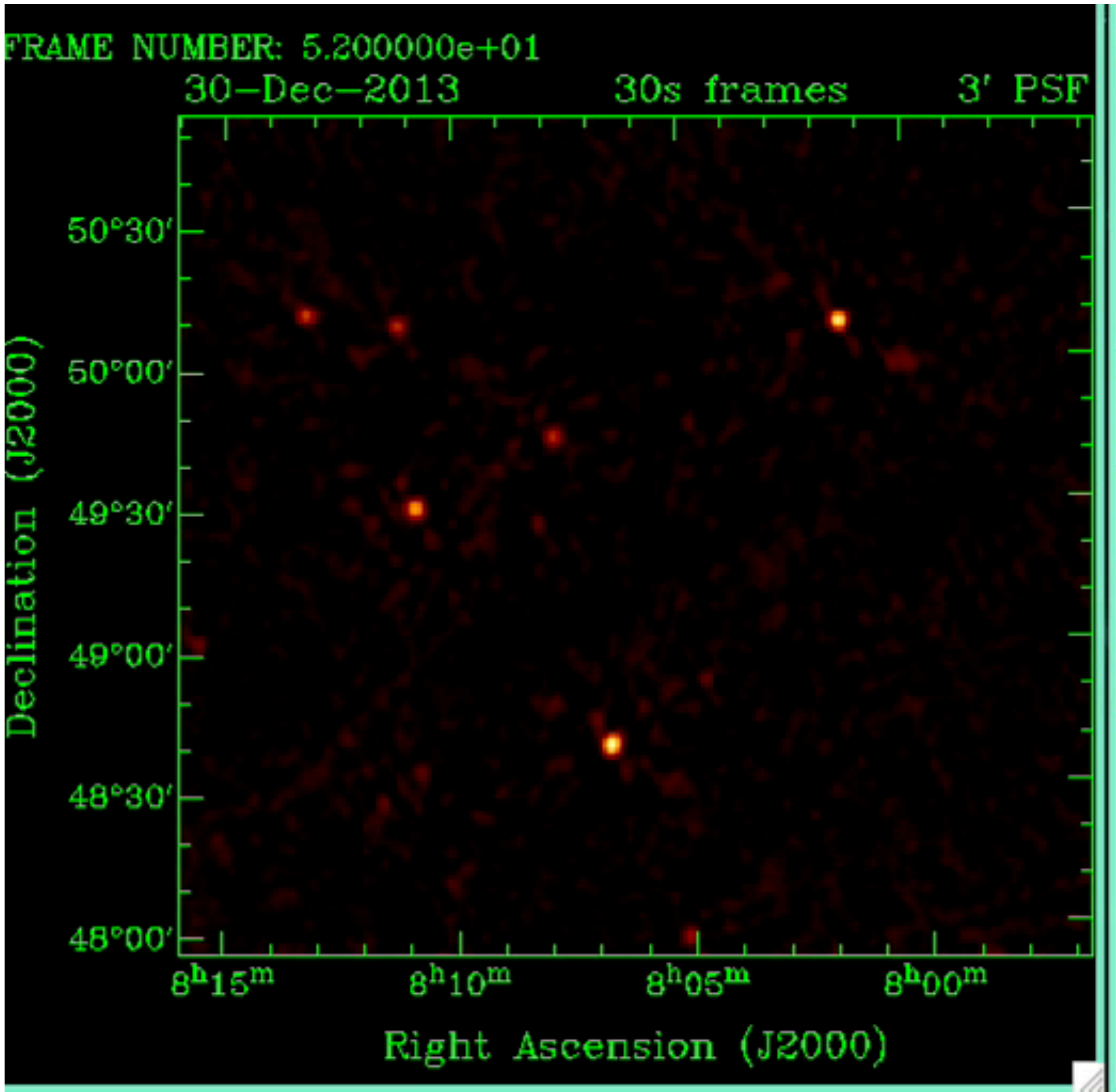
Worked example: What is the cut-off frequency for LOFAR observations carried out when the electron density is $N_e = 2.5 \times 10^5$ cm^{-3} (night time) and $N_e = 1.5 \times 10^6$ cm^{-3} (day time)?

$$\nu_{\rm p}[{\rm Hz}] = 8.97 \times 10^3 \sqrt{\frac{2.5 \times 10^5}{[{\rm cm}^{-3}]}} = 4.5\ {\rm MHz} \qquad ({\rm night\ time})$$

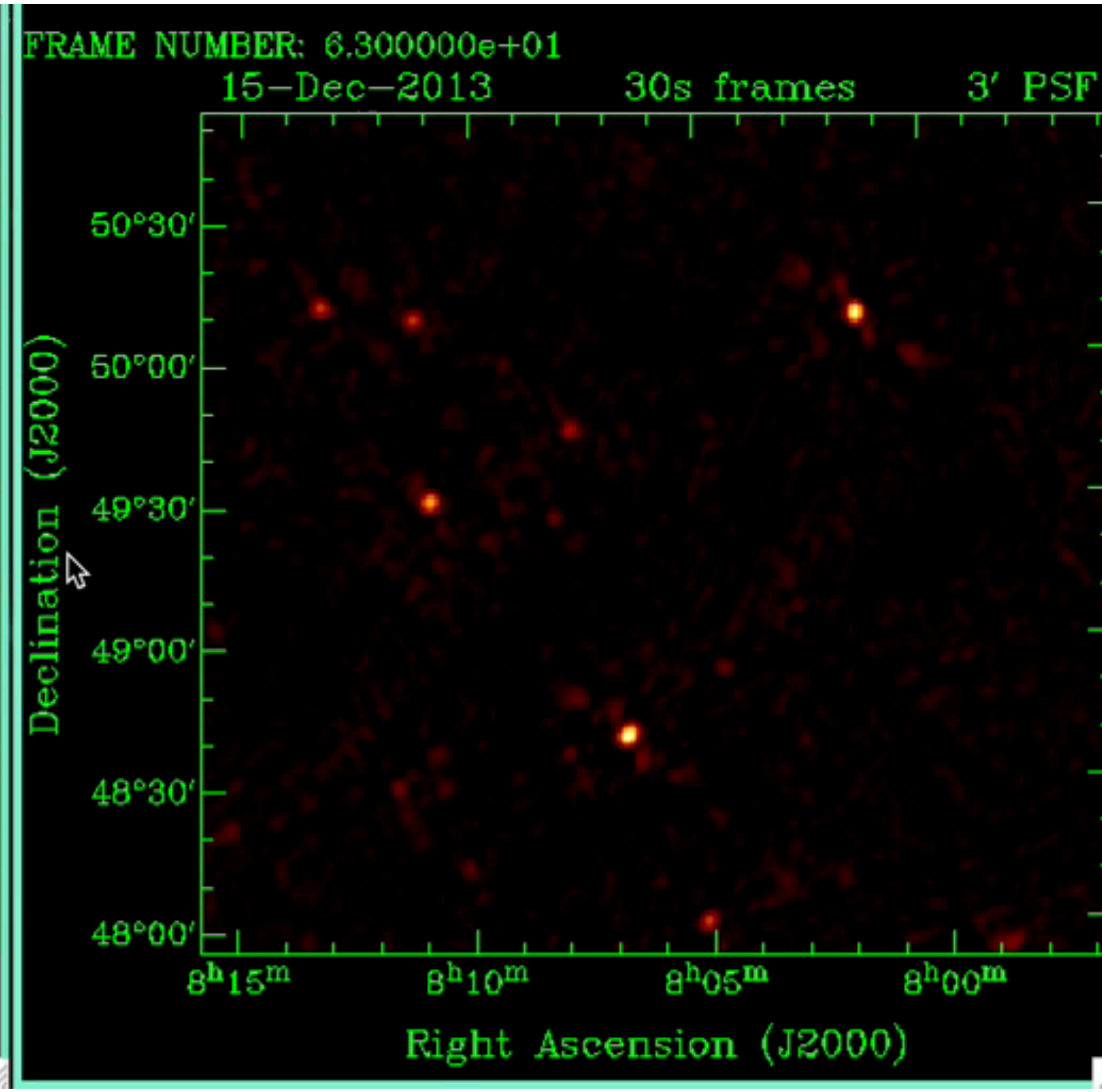
$$\nu_{\rm p}[{\rm Hz}] = 8.97 \times 10^3 \sqrt{\frac{1.5 \times 10^6}{[{\rm cm}^{-3}]}} = 11\ {\rm MHz} \qquad ({\rm day\ time})$$

- At frequencies,
  1. $\omega < \omega_p$: n^2 is negative, reflection (v < 10 MHz),
  2. $\omega > \omega_p$: n^2 is positive, refraction (10 MHz < v < 10 GHz),
  3. $\omega \gg \omega_p$: n^2 is unity (v > 10 GHz).

- Bad observing conditions



- Good observing conditions



1.3 The high-frequency cut-off: Absorption (e-MERLIN, EVN, NOEMA, ALMA)

- Molecules in the atmosphere can absorb the incoming radiation, but also emit radiation (via thermal emission).

- Mass absorption co-efficient *(k)*: From atomic and molecular physics, define for various species, *i*,

$$k_i = \frac{\sigma \, n_i}{r_i \, \rho_0}$$

Cross-section (cm^2)

Number density of particles (cm^{-3})

Mass attenuation co-efficient (cm^2 g^{-1})

Mixing ratio (= ρ_i/ρ_0)

Mass density of air (g cm^{-3})

- Optical depth *(τ)*: A measure of the absorption / scattering (attenuation) of electromagnetic radiation in a medium (probability of an interaction),

$$\tau_i(\lambda, z_0) = \int_{z_0}^{\infty} n_i(z) \, \sigma \, dz = \int_{z_0}^{\infty} r_i(z) \, \rho_0(z) \, k_i(\lambda) \, dz$$

or, in terms of the linear absorption co-efficient *(κ)*,

$$\tau_i(\lambda, z_0) = \int_{z_0}^{\infty} \kappa(\lambda, z) \, dz \qquad \text{where} \quad \kappa(\lambda, z) = k_i(\lambda) \, \rho_i(z)$$

linear absorption co-efficient (cm^{-1})

Mass density of species i (g cm^{-3})

Mass attenuation co-efficient (cm^2 g^{-1})

- The attenuation of an incident ray of intensity I_0, received at altitude z_0, summed over all absorbing species is,

$$I(z_0) = I_0 \exp\left[-\sum_i \tau(\lambda, z_0)\right] = I_0 \exp\left[-\tau(z)\right]$$

Where, for convenience, we consider all species together and define the optical depth as a function of zenith angle, $\tau(z)$.

Worked example: What is the optical depth for sky transparencies of 0.5, 0.1 and 0.01?

Rearrange, in terms of τ, and evaluate, $\tau = -\ln\left(\frac{I(z_0)}{I_0}\right)$

$$\tau_{0.5} = -\ln(0.5) = 0.69$$

$$\tau_{0.1} = -\ln(0.1) = 2.3$$

$$\tau_{0.01} = -\ln(0.01) = 4.6$$

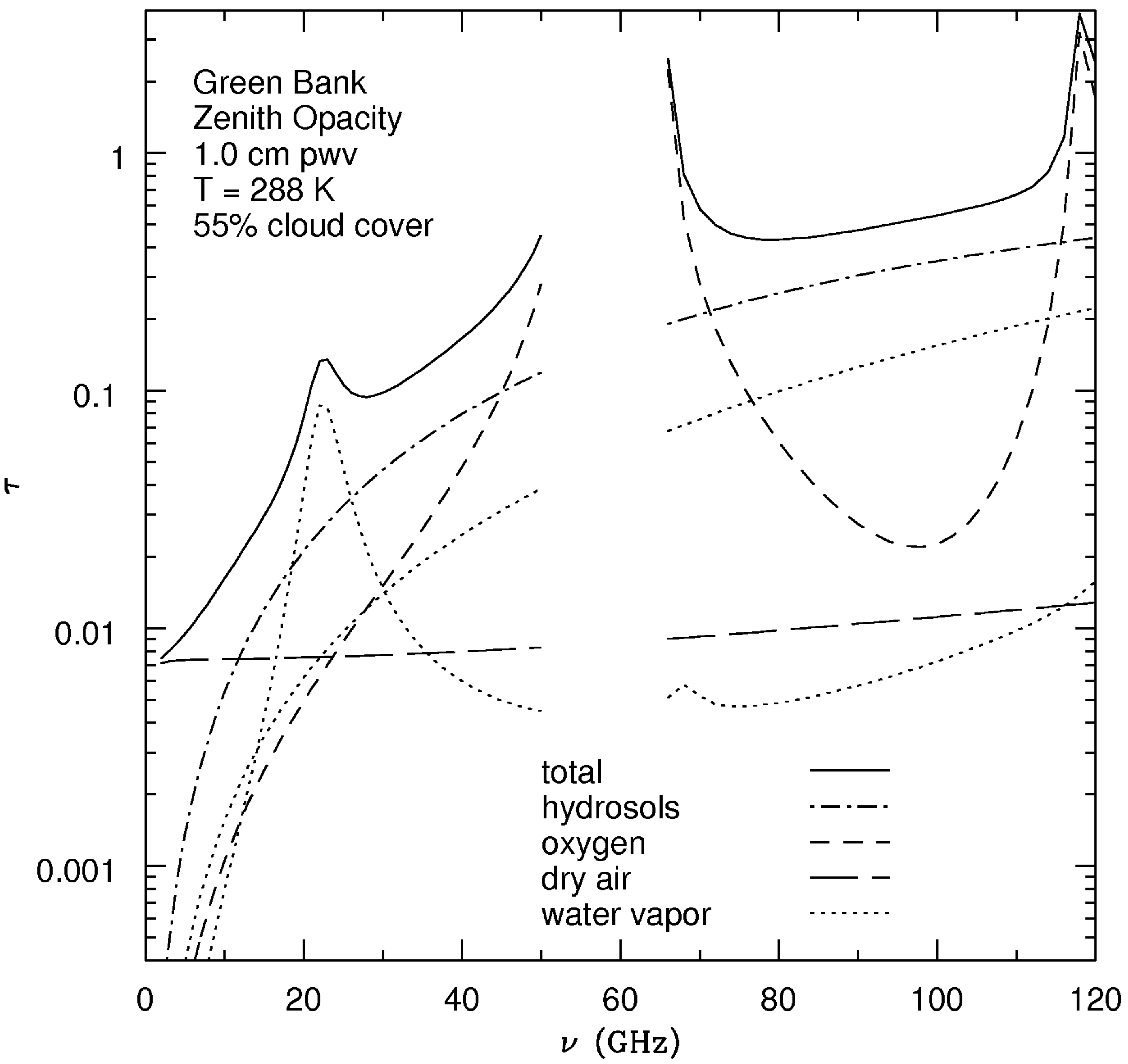
- Note that the opacity changes with the path length, and so depends on the airmass $X(z)$, which assuming a plane parallel atmosphere,

$$\tau(z) = \tau_0 \cdot X(z) \quad \text{where} \quad X(z) = \sec(z)$$

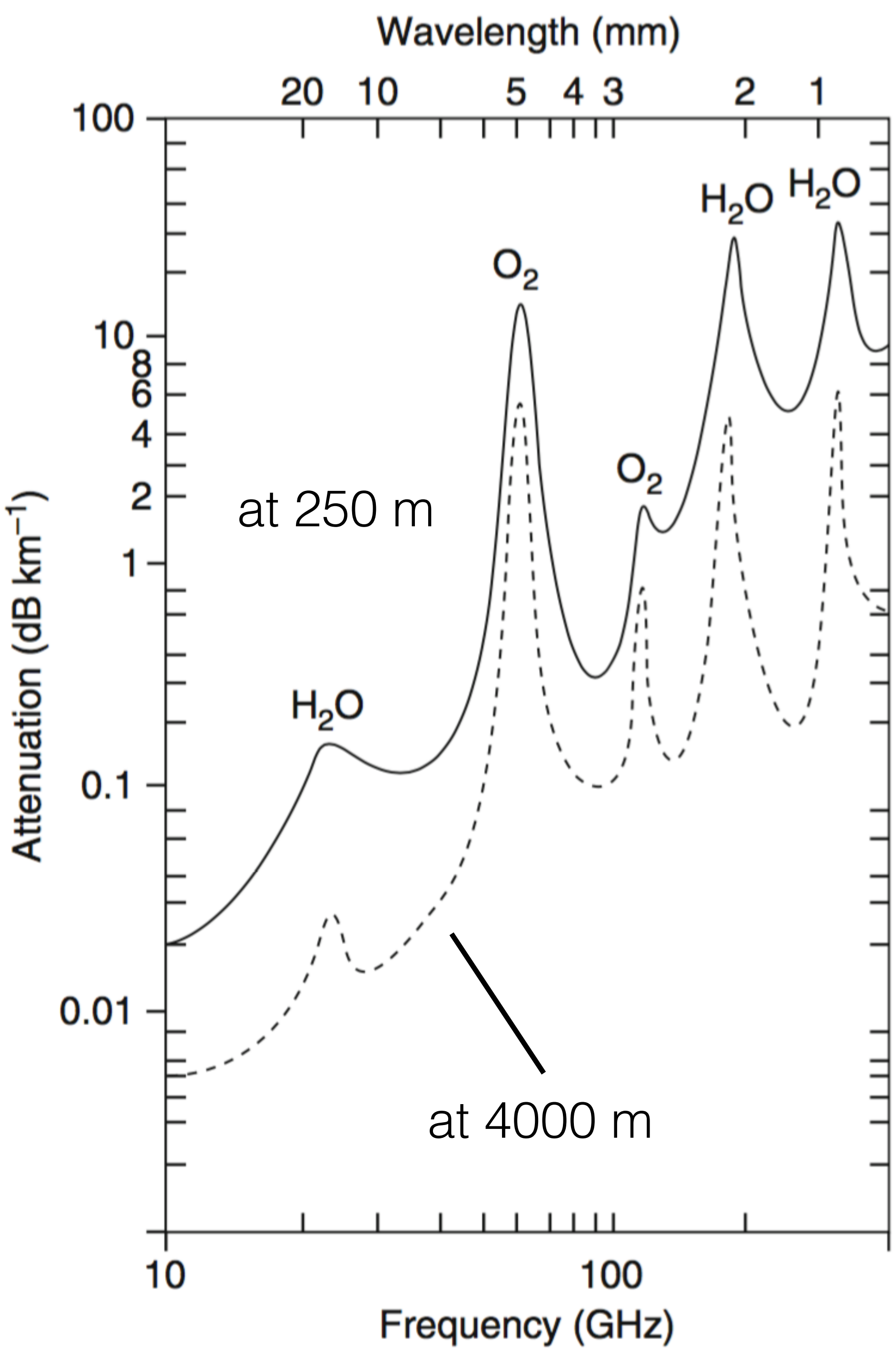
Optical depth at Zenith (τ_0); Airmass ($X(z)$); Zenith angle (z)

- The atmosphere is not completely transparent at radio wavelengths, but $\tau(z)$ varies with frequency ν.

- Zenith opacity is the sum of several component opacities at cm λ.
  - Broadband (continuum) opacity: dry air. $\tau_z \approx 0.01$ and almost independent of ν.
  - Molecular absorption: O_2 has rotational transitions that absorb radio waves and are opaque ($\tau_z \gg 1$) at 52 to 60 GHz.
  - Hydrosols: Water droplets (radius ≤ 0.1 mm) suspended in clouds absorb radiation (proportional to λ^{-2}).
  - Water vapour: Emission line at $\nu \approx 22.235$ GHz is pressure broadened to $\Delta\nu \sim 4$ GHz width + "continuum" absorption from the "line-wings" of very strong H_2O emission at infrared wavelengths (proportional to λ^{-2}).



- The zenith optical depth is dependent on the path length through the material.
  - Higher altitude: Move above the water vapour layer (> 4 km).
  - Drier locations: Move to regions with low water vapour.

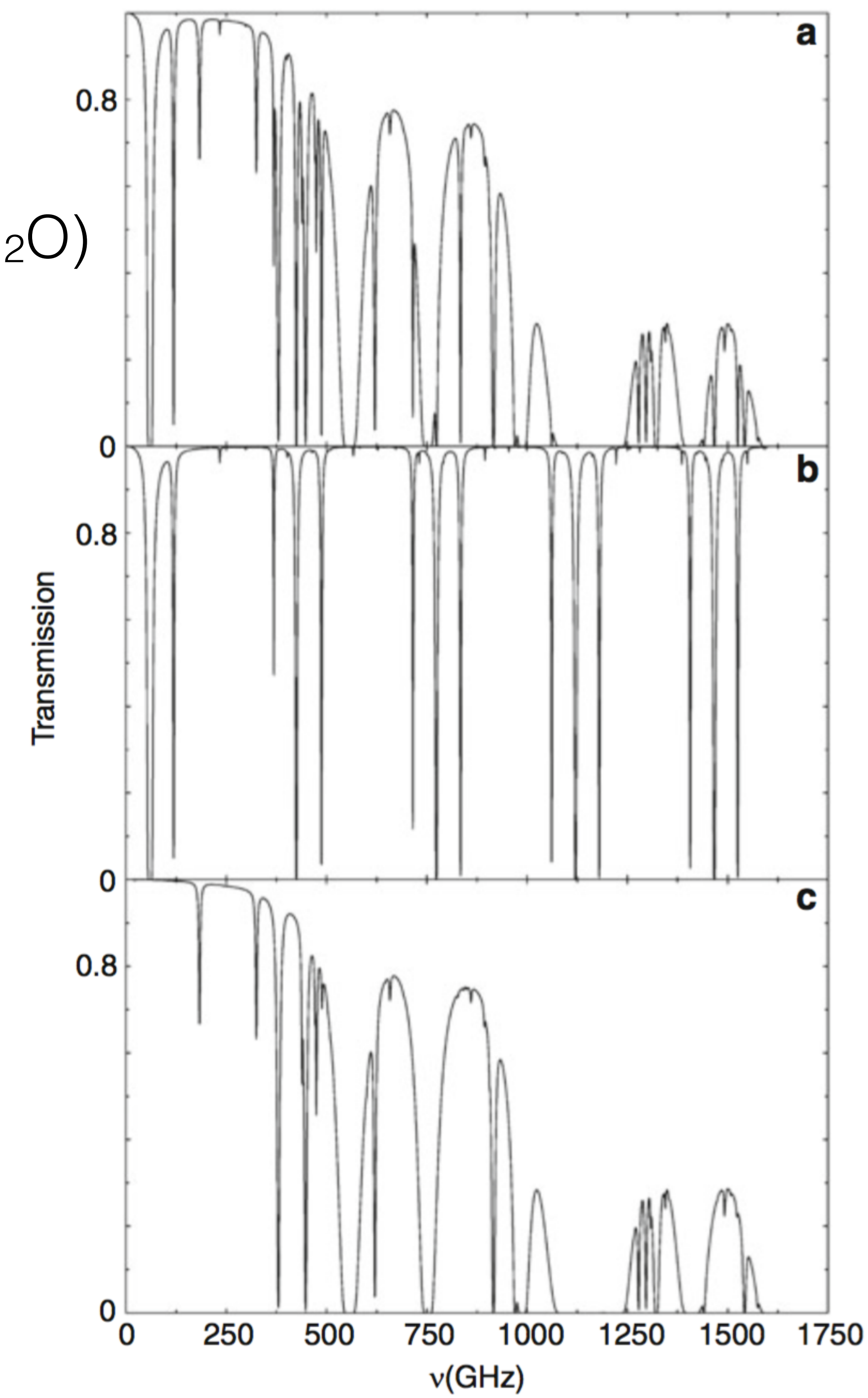


(note τ = 2.3 x Attenuation)

total (O_2 + H_2O)

O_2

H_2O



1.4 The high-frequency cut-off: Emission (e-MERLIN, EVN, NOEMA, ALMA)

- A partially absorbing atmosphere also emits radio noise that can de-grade ground based observations. We can define the total system noise power as an *equivalent noise temperature*,

$$P = \frac{E}{\Delta t} = k\,T\,\Delta\nu$$

in terms of *spectral power*,

$$P_\nu = k\,T_{\rm sys}$$

Spectral power (W Hz^{-1})

System temperature (Receivers; Sky, Ground; etc)

Boltzmann constant = 1.38 x 10^{-23} m^2 kg s^{-2} K^{-1}

where,

$$T_{\rm sys} = T_{\rm bg} + T_{\rm sky} + T_{\rm spill} + T_{\rm loss} + T_{\rm cal} + T_{\rm rx}$$

Noise from Radio background (Galaxy, CMB, etc)

Noise from atmospheric emission

Noise from ground emission

Noise from losses at receiver

Noise from injected noise

Noise from the receiver (Dominates)

- The contribution from the sky opacity to the sky temperature is,

$$T_{\rm sky} = T_{\rm atm}\left[1 - \exp(-\tau_\nu)\right]$$

Emitted sky temperature (K)

Atmospheric kinetic temperature (≡ 300 K)

Optical depth

- Don't want $T_{\rm sky}$ to dominate our noise budget, need to minimise $T_{\rm atm}$ and τ_ν by observing in cold and dry locations (winter; high alt), especially at high frequencies.

Worked example: Using the total opacity data for the Green Bank Telescope (West Virginia; USA; 2800 m) and $T_{\rm atm}$ = 288 K, what is $T_{\rm sky}$ at ν = 5 GHz, 22 GHz and 115 GHz?

How does this compare with the typical receiver temperature, T_{rx} ~ 20 K?

- At ν = 5 GHz, $\tau_z \sim 0.007$, $T_{\rm sky} = 288\left[1 - \exp(-0.007)\right] \sim 2$ K (Good)
- At ν = 22 GHz, $\tau_z \sim 0.15$, $T_{\rm sky} = 288\left[1 - \exp(-0.15)\right] \sim 40$ K (Bad)
- At ν = 115 GHz, $\tau_z \sim 0.8$, $T_{\rm sky} = 288\left[1 - \exp(-0.8)\right] \sim 160$ K (Bad)

Key concept: The partially transparent atmosphere allows radio waves to be detected from ground-based telescopes, but also attenuates the signal due to absorption / scattering, and also adds noise to the measured signal.

1.5 Early Radio Astronomy

- The first detection of radiation at radio wavelengths was not made until 1932 due to,
  i) limitations of technology (our eyes), but then the communication era started,
  ii) the expectation that celestial objects would be too faint.

- The spectral brightness B_ν at frequency ν of a blackbody object (stars) is given by Planck's law.

Spectral brightness (W m^{-2} Hz^{-1} sr^{-1})

Planck constant = 6.626 x 10^{-34} m^2 kg s^{-1}

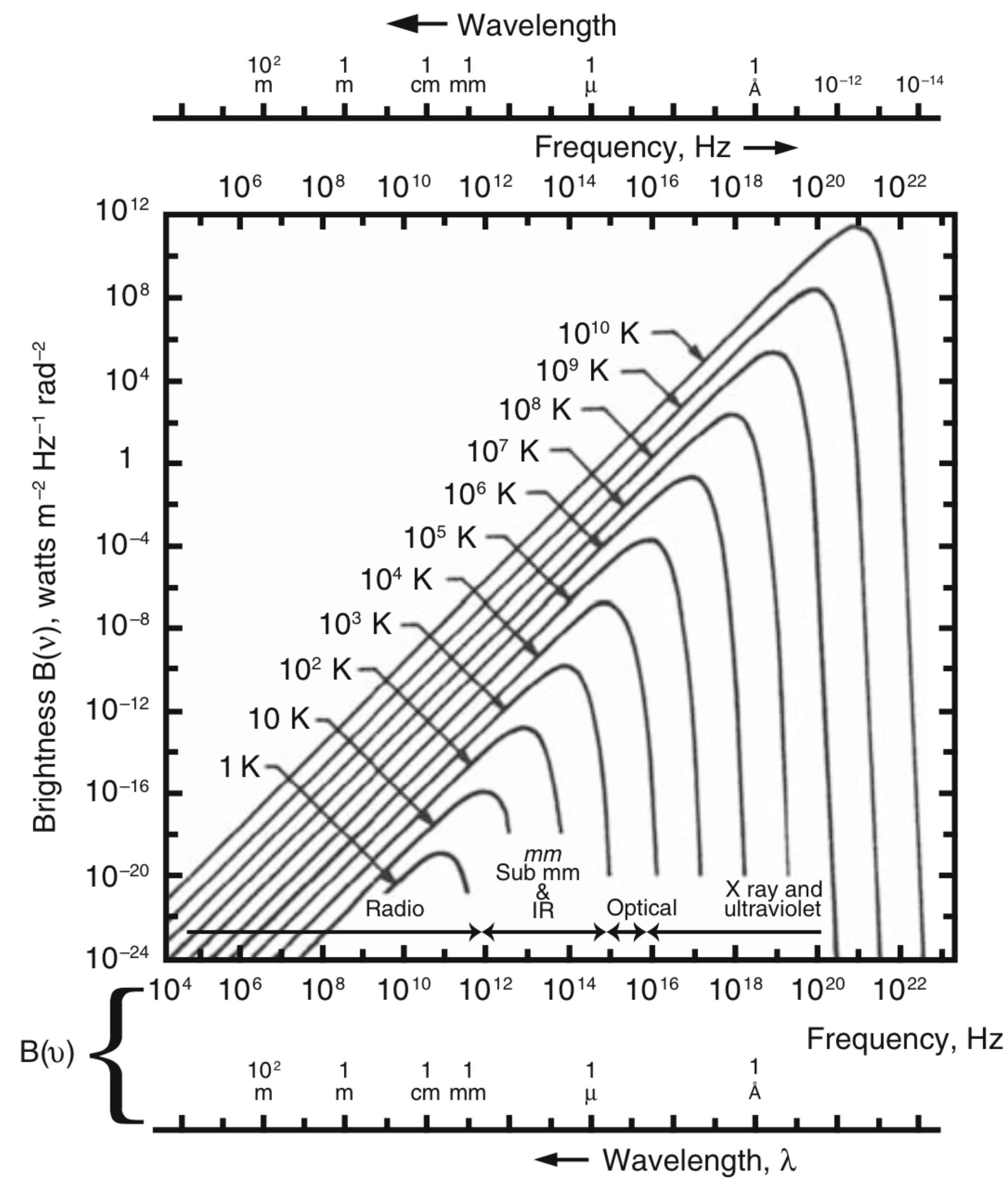
$$B_\nu(T) = \frac{2h\nu^3}{c^2}\frac{1}{\exp\frac{h\nu}{kT} - 1}$$

Speed of light constant = 3 x 10^8 m s^{-1}

Absolute temperature (K)

- In the low frequency radio limit, $h\nu / kT \ll 1$.

$$B_\nu(T) \approx \frac{2h\nu^3}{c^2}\frac{kT}{h\nu} = \frac{2kT\nu^2}{c^2} = \frac{2kT}{\lambda^2}$$

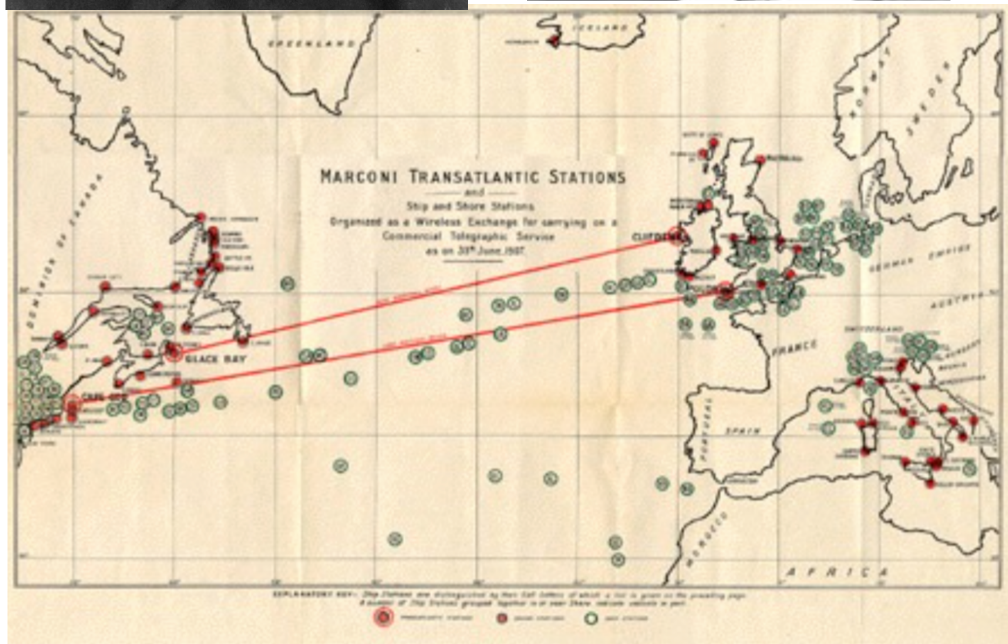


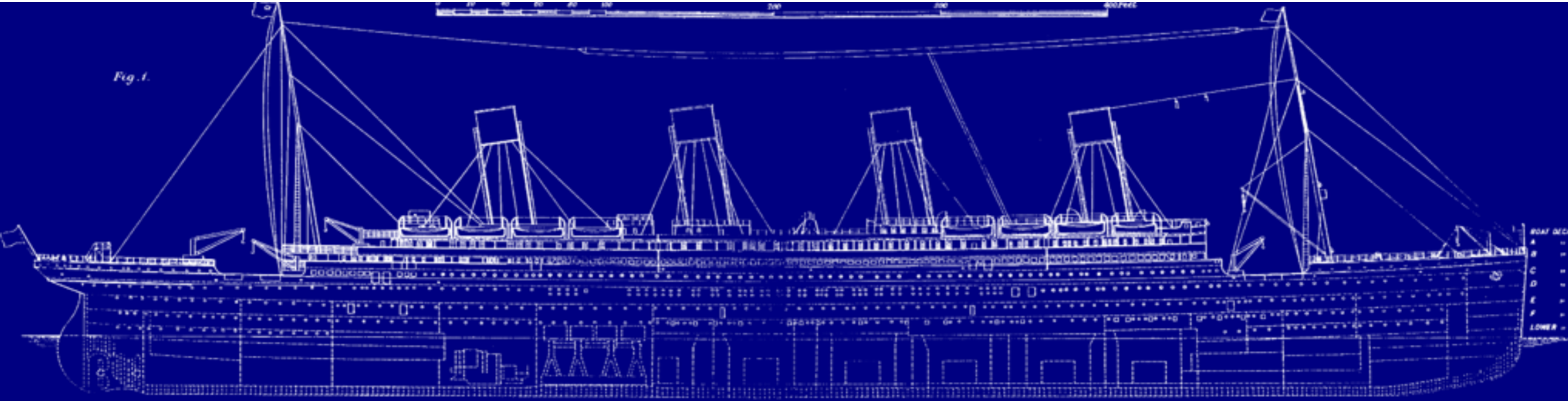
- Long distance communication developed by Marconi & Ferdinand Braun - Nobel Prize 1909

Evolution of frequency over the years

- pre-1920: <100 kHz.
- ca. 1920: shift to 1.5 MHz.
- post-1920: 10s of MHz (more voice channels, less effected by the ionosphere and thunderstorms).

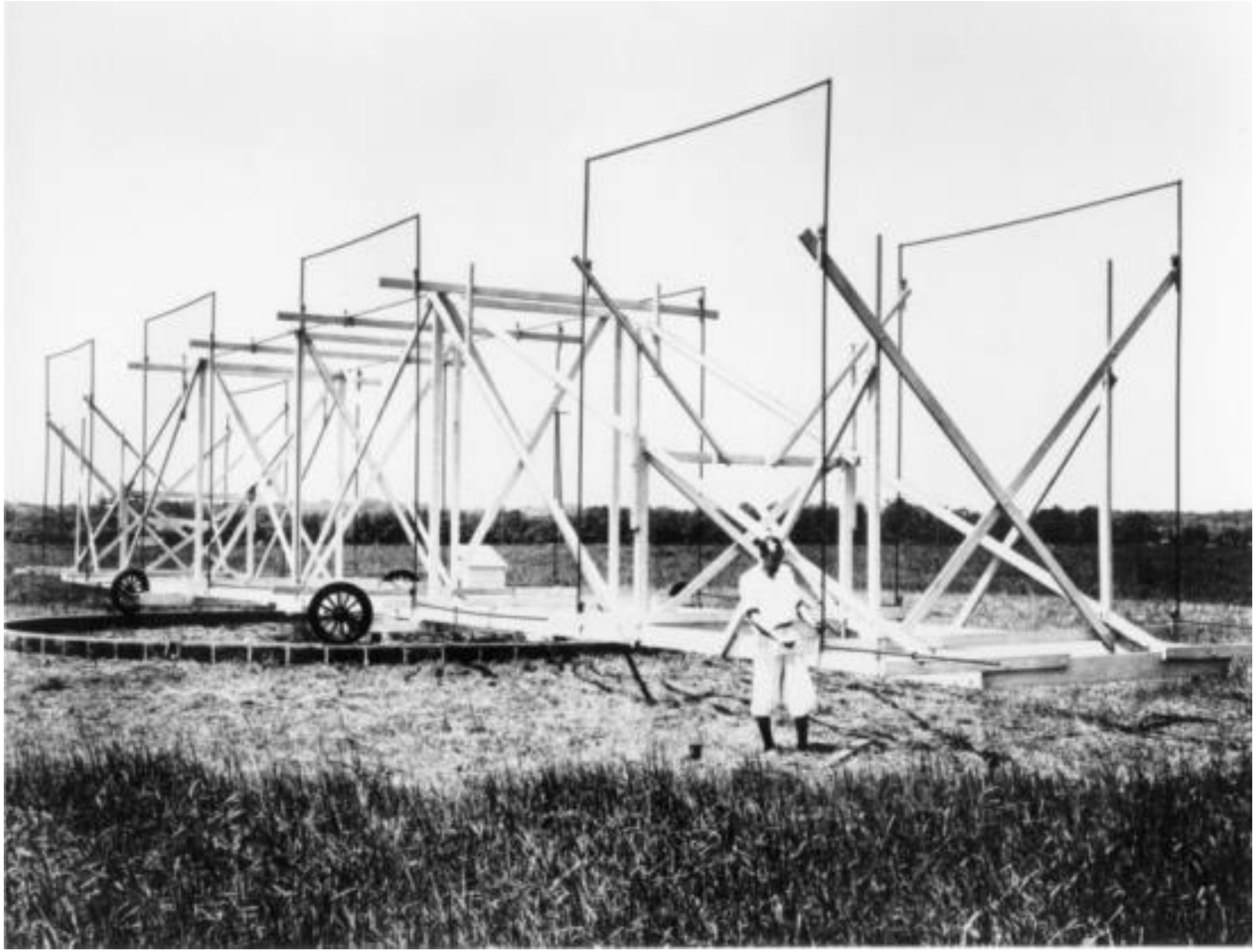
- Research labs sprung up in early-1900s

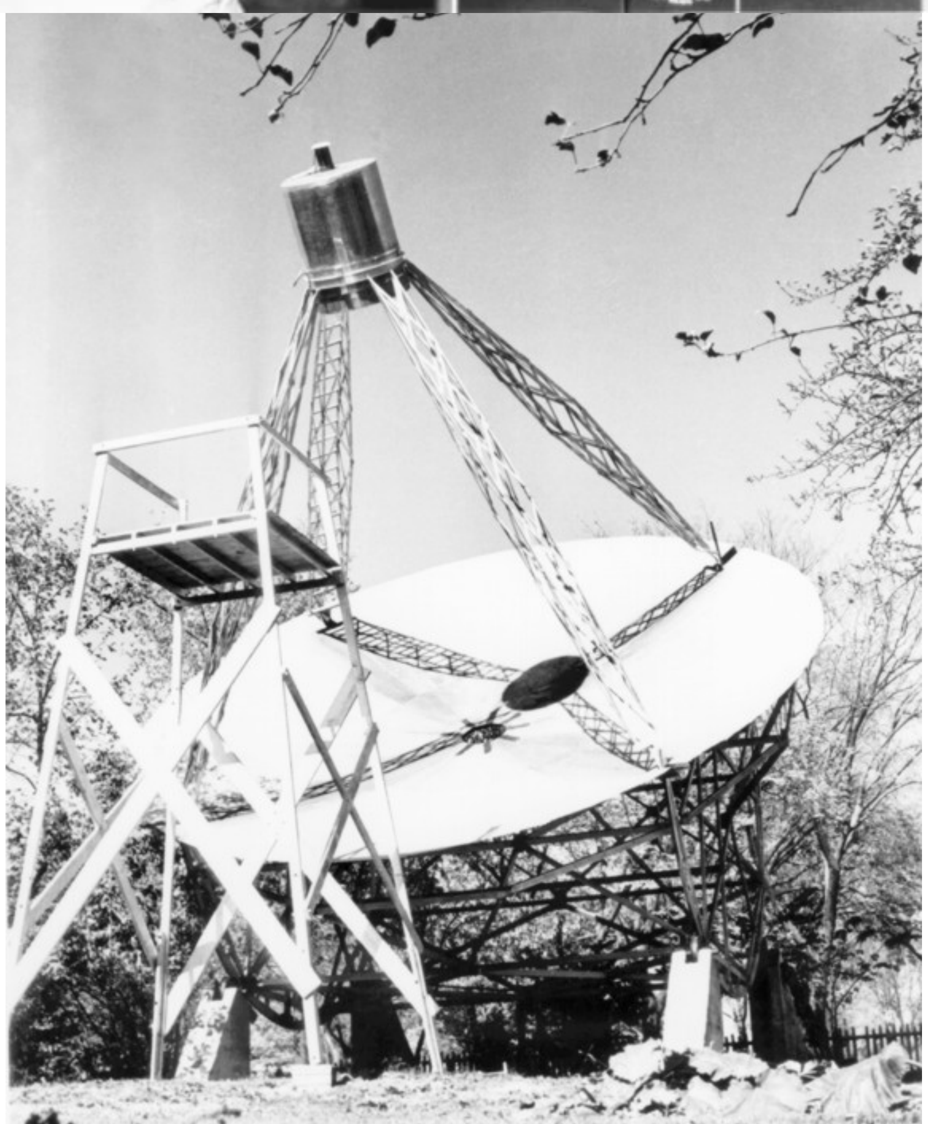




- Karl Jansky (1933, published) discovered a radio signal at 20.5 MHz that varied steady every 23 hours and 56 minutes (Sidereal day).

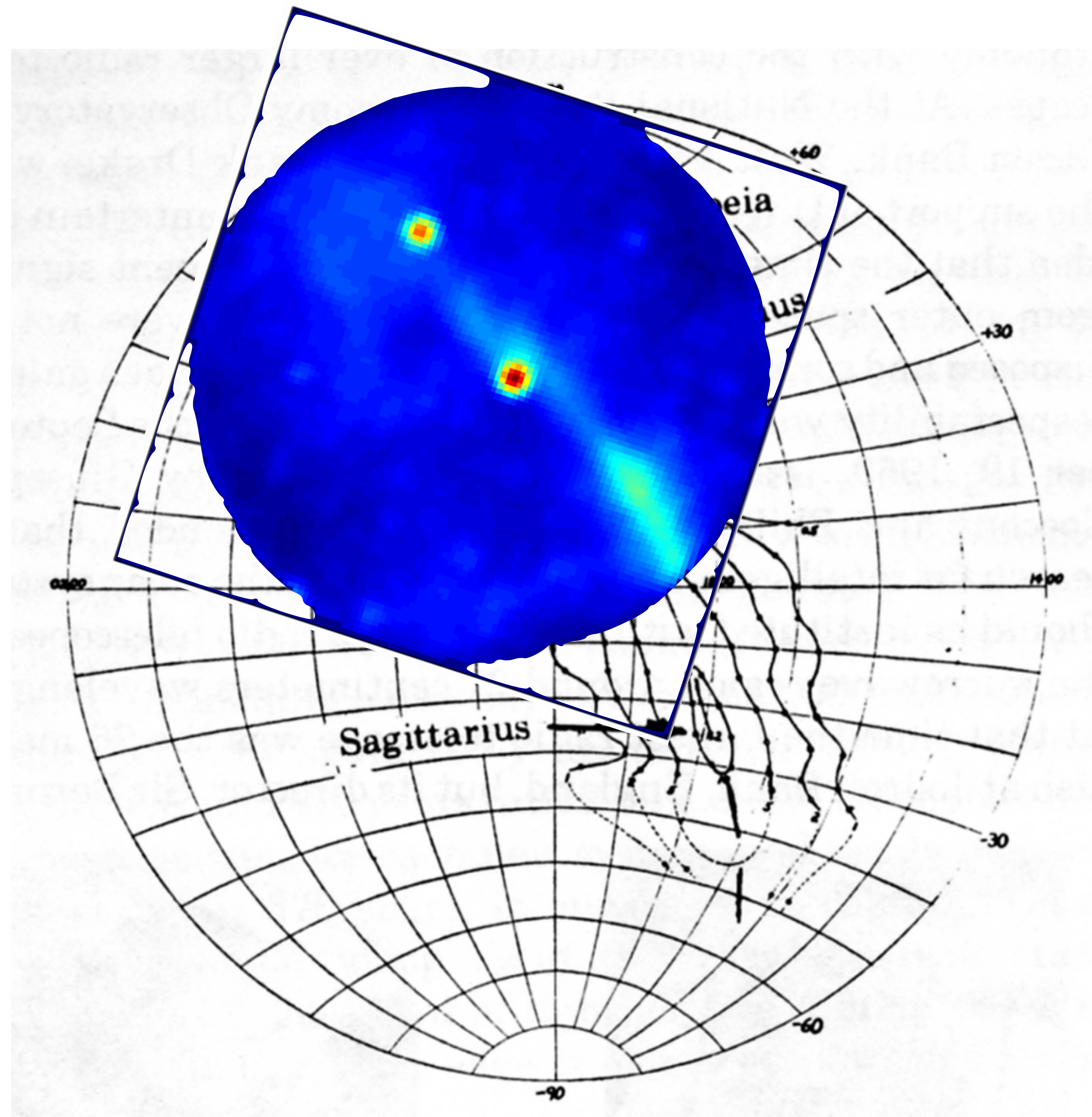
"The data give for the co-ordinates of the region from which the disturbance comes, a right ascension of 18 hours and declination -10 degrees." He had detected the Galactic Centre.

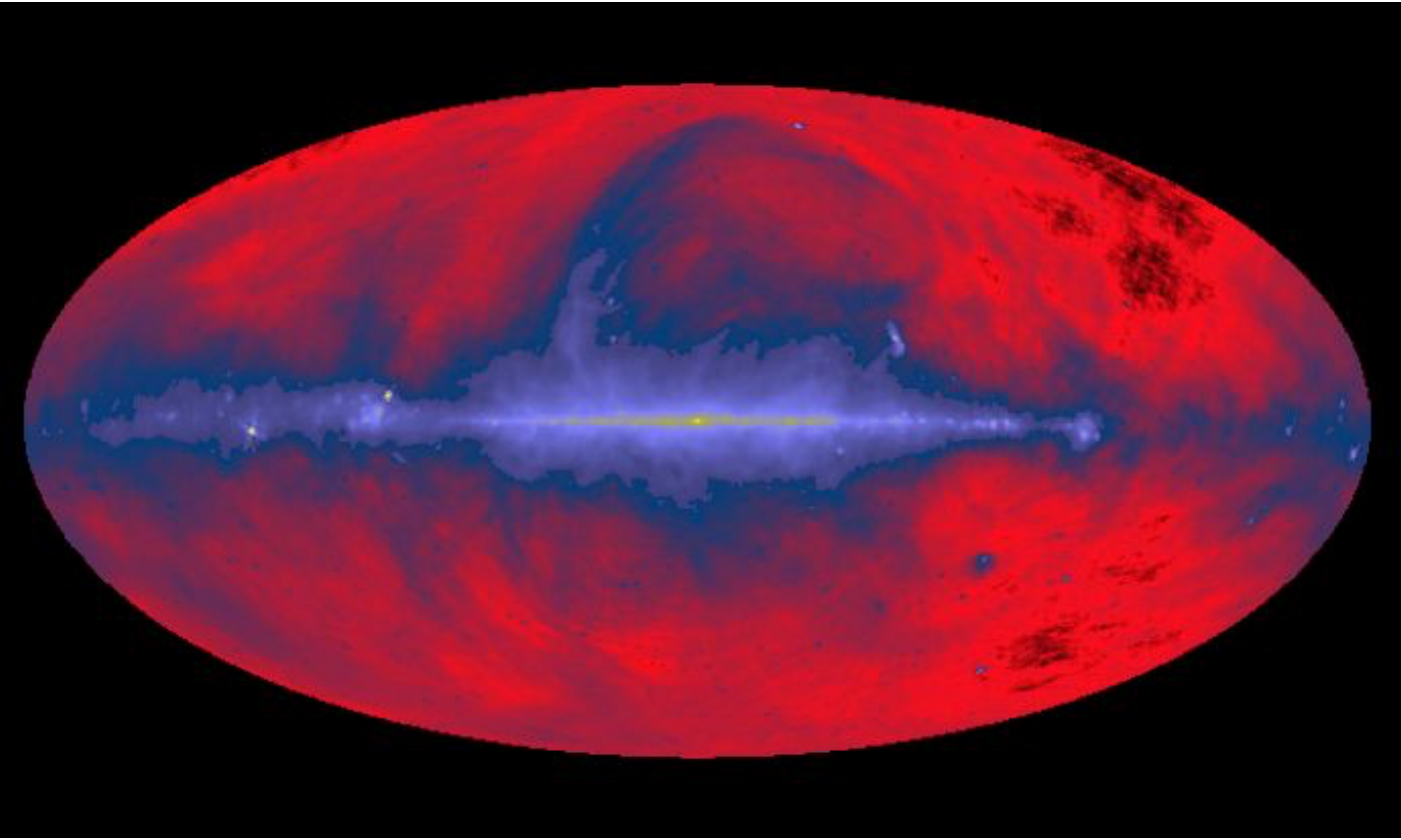


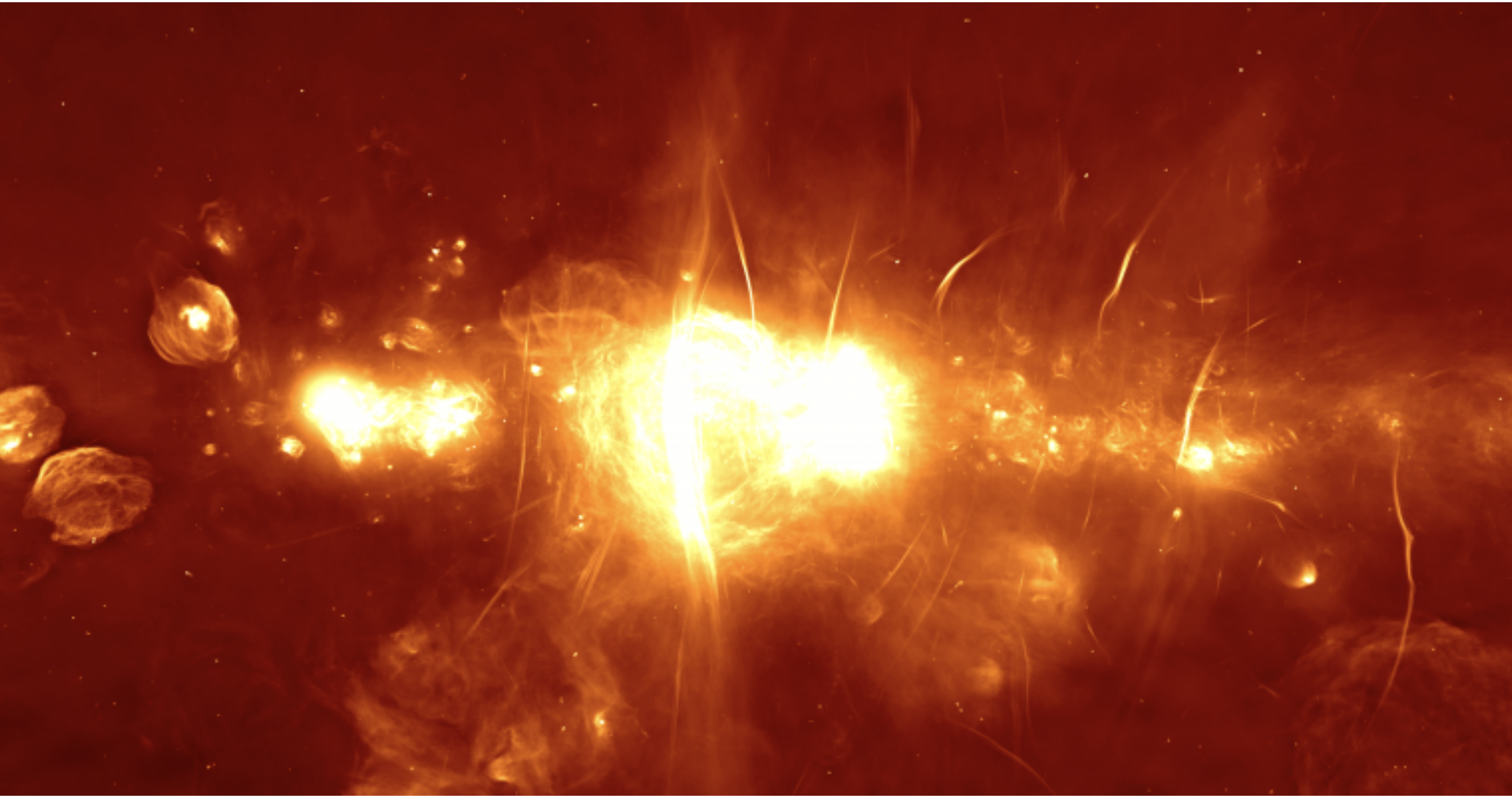


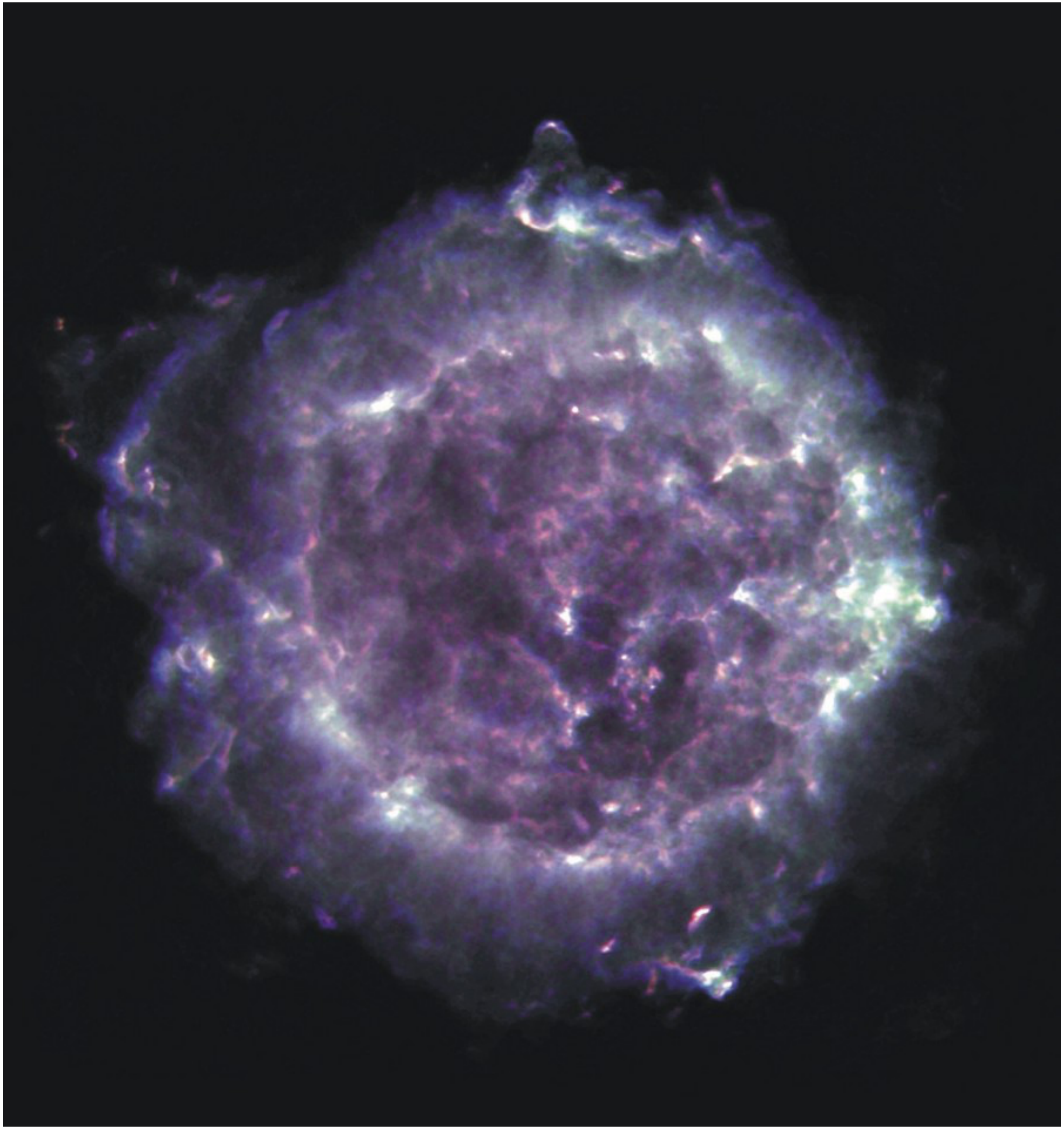
- Grote Reber (1937-39), using his own 10 m telescope, made no detection at 3300 and 910 MHz, ruling out a Planck spectrum (B_ν propto ν^2).

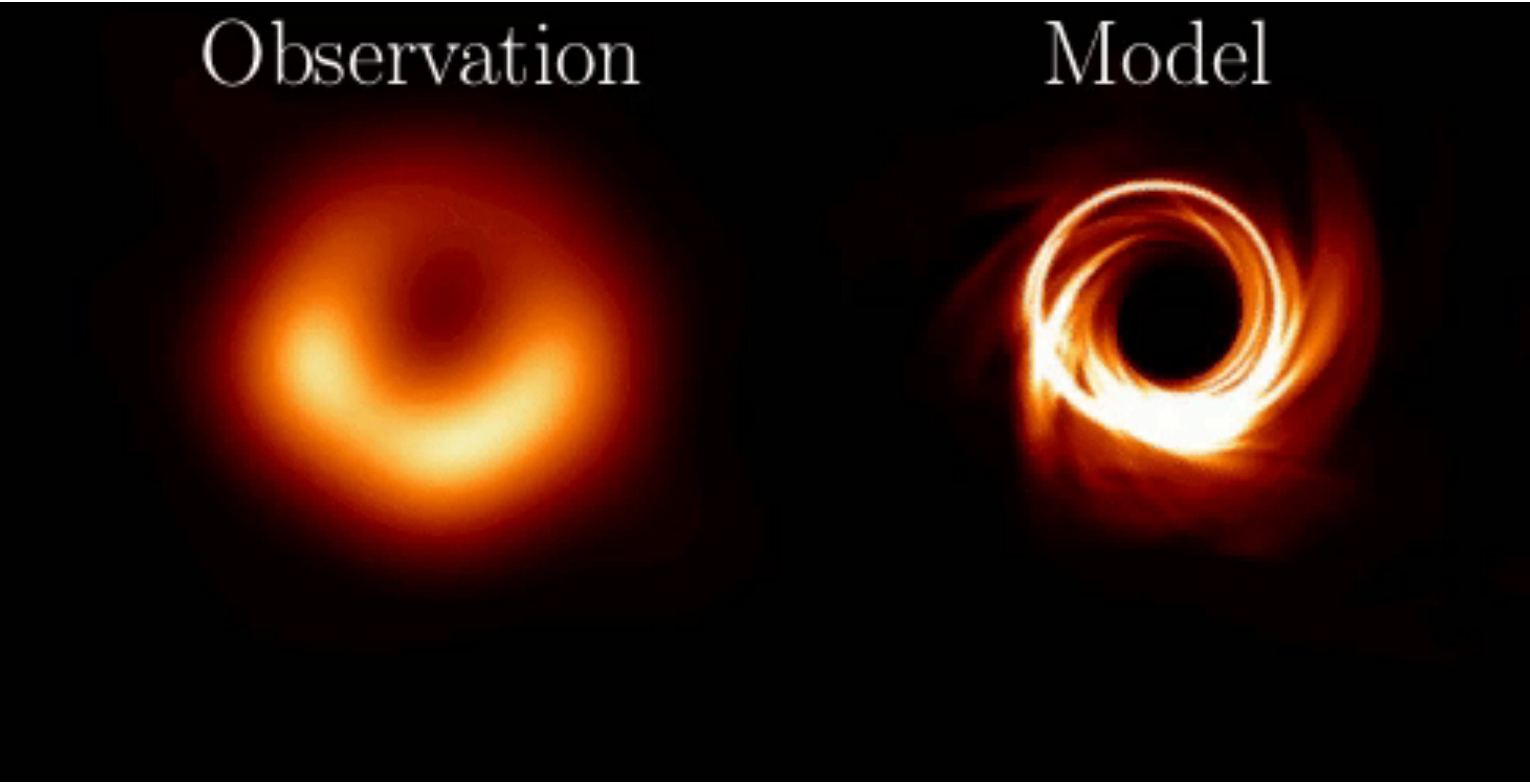
- Detection made at 150 MHz, confirming Jansky's result and finding the spectrum must be non-thermal.

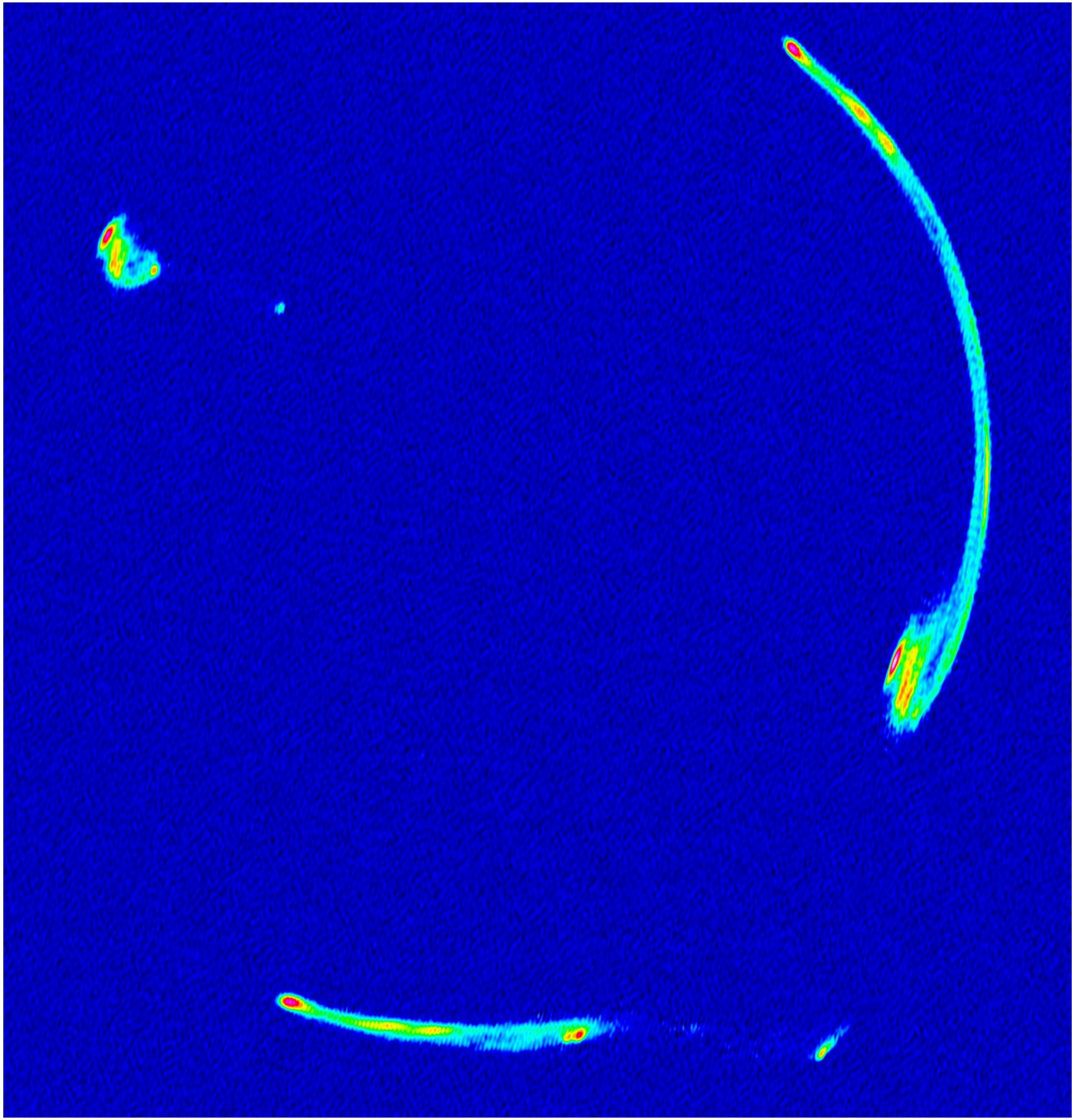


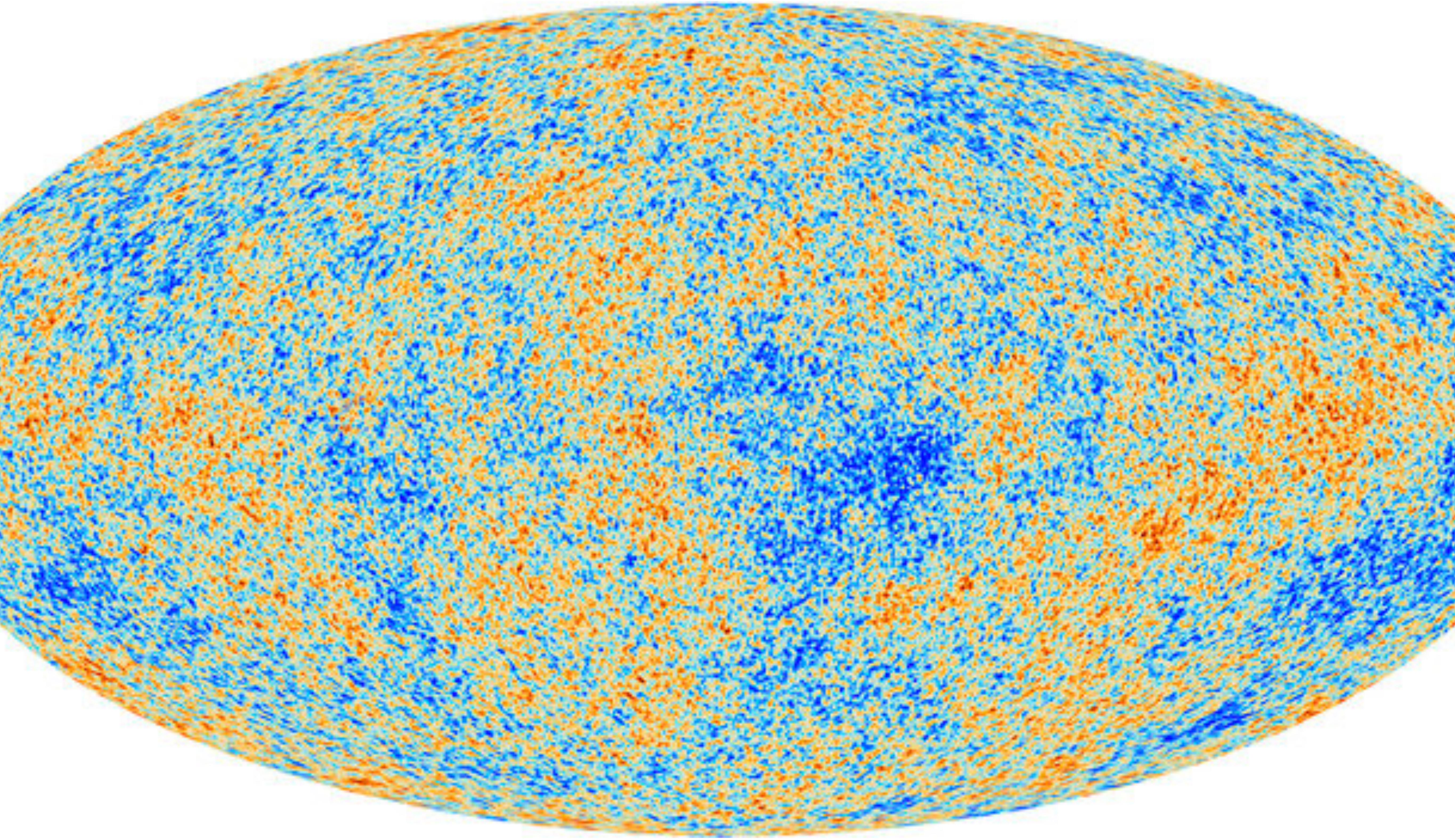


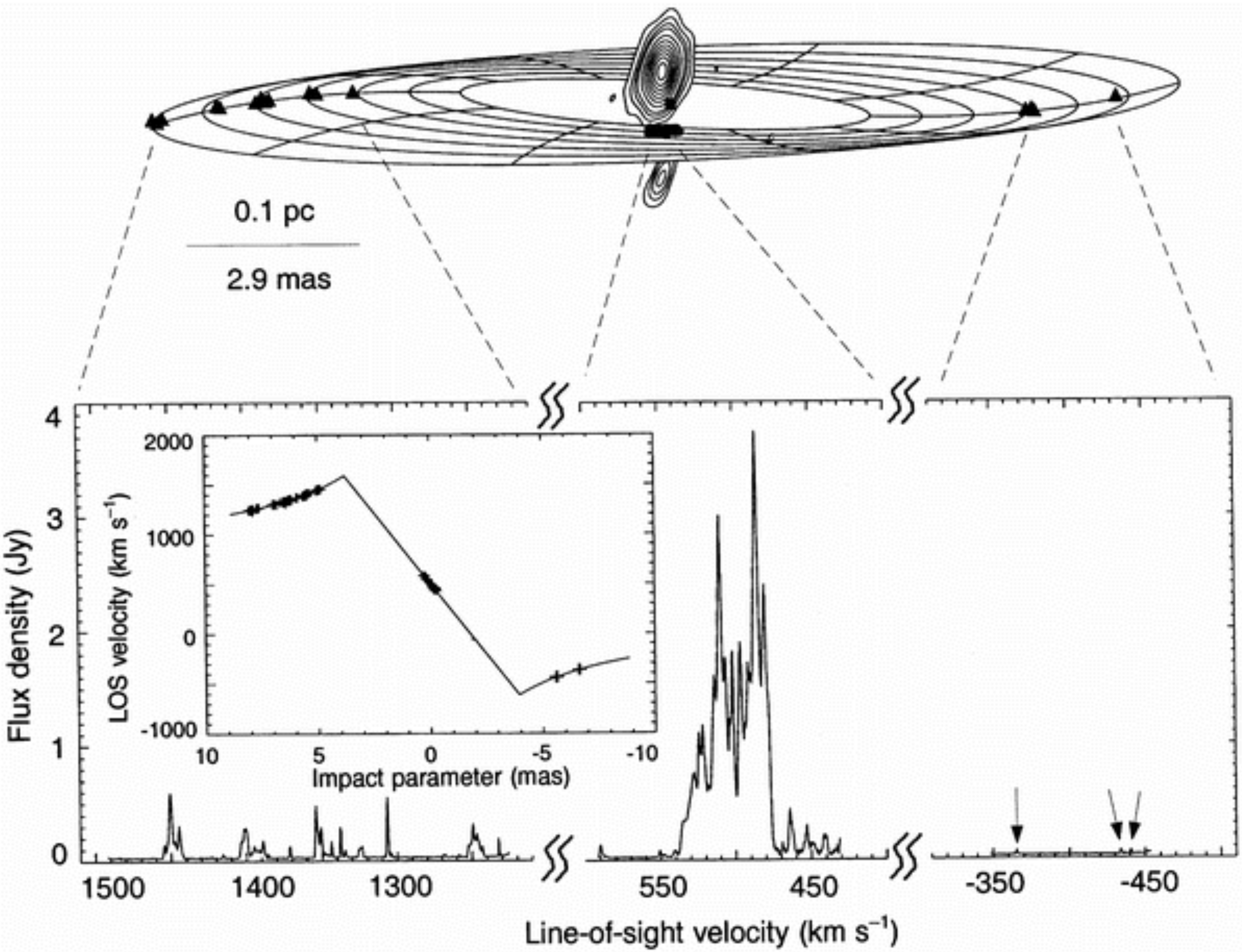





Observation
Model





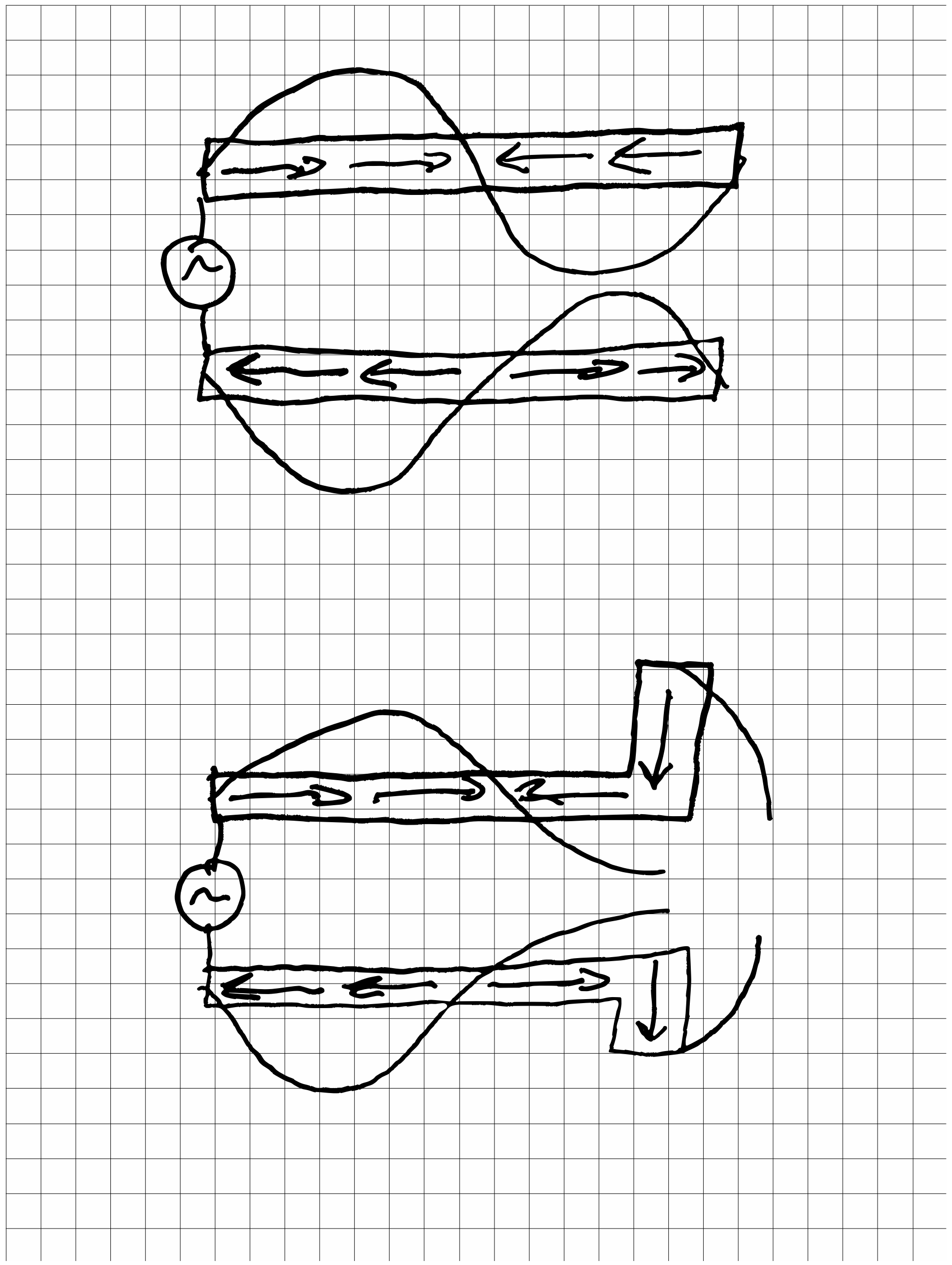


0.1 pc
2.9 mas
Flux density (Jy)
4
3
2
1
0
1500
1400
1300
550
450
-350
-450
Line-of-sight velocity (km s^{-1})
LOS velocity (km s^{-1})
2000
1000
0
-1000
10
5
0
-5
-10
Impact parameter (mas)

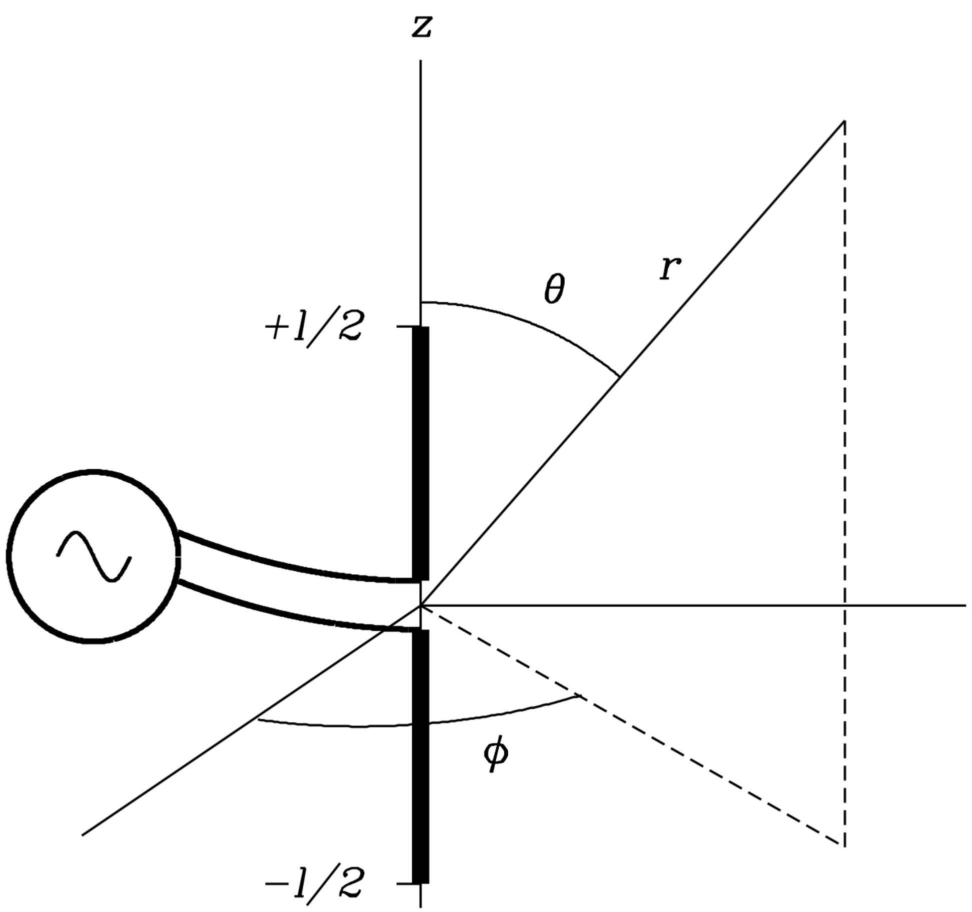
2. The response of a dipole antenna (LOFAR)

2.1 Dipole antenna fundamentals

- Antenna: A device for converting electromagnetic radiation in space into electrical currents (transmitting and receiving).



- Consider a simple thin-wire transition line antenna of length λ. The current along both wires is out of phase.
- By bending the edges of the transmission line (l < λ / 10), the current is now is phase, but there is a build up of charge at the ends (dipole).
- When the length is λ / 2 (or multiple), the current is a maximum at the antenna feed.



Consider a Hertzian small ($l \ll \lambda$) dipole transmitter (same as for a receiving dipole, but easier to understand).

Two co-linear conductors (e.g. wires, conducting rods), driven by a current source at the gap. The driving current I is a time varying sinusoidally with angular frequency,

$$\omega = 2\pi\nu$$

$$I = I_0 \cos(\omega t) = I_0 e^{-i\omega t}$$

(Only consider the real part of $e^{-i\omega t} = \cos(\omega t) + i \sin(\omega t)$)

The time varying current density is defined as, $J = \frac{I}{q} = \frac{I_0}{q} e^{-i\omega t}$ inside the dipole,

and $J = 0$ outside the dipole.

- We want to measure the power radiated from such an antenna, so we calculate,
    1. The electromagnetic vector potential *A,*
    2. The magnetic field induction *B*, and hence the magnetic field intensity *H,*
    3. The electric field intensity *E,*
    4. The Poynting flux *S*

1. The electromagnetic vector potential

The induced magnetic field *B* is related to the vector potential by,

$$\vec{B} = \nabla \times \vec{A}$$

where,

$$\vec{A}(x) = \frac{\mu_0}{4\pi} \int\int\int \vec{J}(x) \frac{e^{ik|x-x'|}}{|x-x'|} d^3x'$$

i.e., the integral of the current density over the volume of the dipole (*dV* = *q dz*).

The current runs from z = −Δl / 2 and z = +Δl / 2 along the z-axis, thus

$\vec{J}_x = 0$ and $\vec{A}_x = 0$

$\vec{J}_y = 0$ and $\vec{A}_y = 0$

only $\vec{J}_z = \frac{I}{q} e^{-i\omega t}$ is non-zero.

Therefore, our vector potential becomes,

$$\vec{A}_z = \frac{\mu_0}{4\pi} \int_{-\Delta l/2}^{+\Delta l/2} \frac{I(z)}{q}\, e^{-i\omega t}\, \frac{e^{ikr}}{r}\, q\, dz$$

$$= \frac{\mu_0}{4\pi} \frac{e^{-i(\omega t - kr)}}{r} \int_{-\Delta l/2}^{+\Delta l/2} I(z)\, dz$$

If the current is constant,

$$\int_{-\Delta l/2}^{+\Delta l/2} I(z)\, dz = I\, [z]_{-\Delta l/2}^{+\Delta l/2} = I\, \Delta l$$

Therefore, our vector potential for a constant current is,

$$\boxed{\vec{A}_z = \frac{\mu_0}{4\pi} \frac{e^{-i(\omega t - kr)}}{r}\, I \Delta l}$$

2. The magnetic induction is related to the magnetic vector potential via,

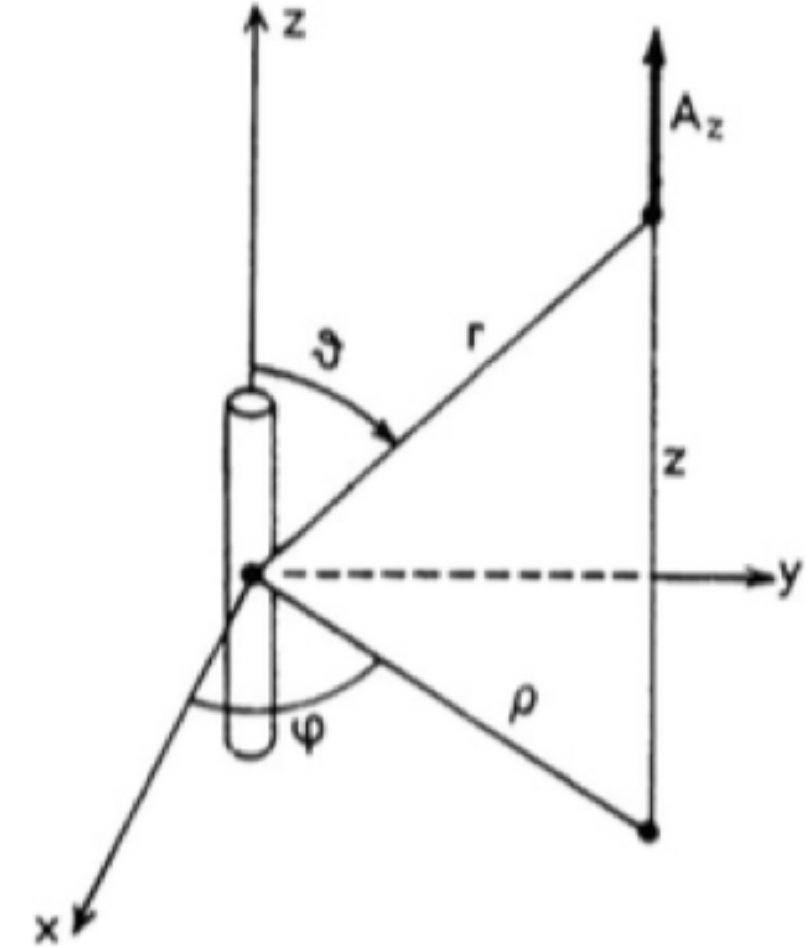
$$\vec{B} = \nabla \times \vec{A}$$

We can de-compose the curl of A into three orthogonal cylindrical co-ordinates (*ρ, ψ, z*), using standard definitions,

$$(\nabla \times \vec{A})_\rho = \frac{1}{\rho}\frac{\partial A_z}{\partial \psi} - \frac{\partial A_\psi}{\partial z}$$

$$(\nabla \times \vec{A})_\psi = \frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}$$

$$(\nabla \times \vec{A})_z = \frac{1}{\rho}\left(\frac{\partial(\rho A_\psi)}{\partial \rho} - \frac{\partial A_\rho}{\partial \psi}\right)$$



As $A_\rho = A_\psi = 0$, the B-field must be perpendicular to the vector potential (A_z).

For simplicity lets evaluate,

$$B_\psi = (\nabla \times \vec{A})_\psi = \frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho} = -\frac{\partial A_z}{\partial \rho} = -\frac{\partial A_z}{\partial r}\frac{\partial r}{\partial \rho}$$

In the cylindrical system,

$$r^2 = \rho^2 + z^2 \qquad r = (\rho^2 + z^2)^{1/2}$$

$$\frac{\partial r}{\partial \rho} = \frac{1}{2}(\rho^2 + z^2)^{-1/2}\, 2\rho = \frac{\rho}{r} = \sin\theta$$

Next,

$$\frac{\partial A_z}{\partial r} = \frac{\mu_0}{4\pi}\, I\Delta l\, e^{-i\omega t}\, \frac{\partial}{\partial r}\left[\frac{e^{ikr}}{r}\right]$$

We solve this using the quotient rule,

$$\left[\frac{u(r)}{v(r)}\right] = \frac{u'(r)v(r) - v'(r)u(r)}{v(r)^2}$$

$$u(r) = e^{ikr} \qquad v(r) = r$$

$$u'(r) = ik\, e^{ikr} \qquad v'(r) = 1$$

$$\frac{\partial}{\partial r}\left[\frac{e^{ikr}}{r}\right] = \frac{ik\, e^{ikr}\cdot r - 1\cdot e^{ikr}}{r^2} = \frac{(ikr-1)e^{ikr}}{r^2}$$

Therefore our *B*-field in the *ψ* direction becomes,

$$B_\psi = -\frac{\partial A_z}{\partial r}\frac{\partial r}{\partial \rho} = -i\,k\,\frac{\mu_0}{4\pi}\, I\Delta l\, \frac{\sin\theta}{r}\left(1 - \frac{1}{ikr}\right) e^{-i(\omega t - kr)}$$

Since,

$$k = \frac{2\pi}{\lambda}$$

$$B_\psi = -i\,\mu_0\, \frac{I\Delta l}{2\lambda}\, \frac{\sin\theta}{r}\left(1 - \frac{1}{ikr}\right) e^{-i(\omega t - kr)}$$

which, from the materials equations, gives for the magnetic field intensity,

$$B = \mu_0 H$$

$$H_\psi = -i \frac{I\Delta l}{2\lambda} \frac{\sin\theta}{r} \left(1 - \frac{1}{ikr}\right) e^{-i(\omega t - kr)}$$

Again, the magnetic field intensity is perpendicular to the vector potential, that is, perpendicular to the element.

3. Now, let's consider the electric field intensity. From Maxwell's equations,

$$\nabla \times \vec{H} = \vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

which, because we are away from the current element ($J = 0$), simplifies to,

$$\nabla \times \vec{H} = \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

We are dealing with harmonic waves of the form,

$$E(r,t) = E_0 e^{-i(wt-kr)}$$

$$\dot{E}(r,t) = E_0 e^{-i(wt-kr)} \cdot -i\omega = -i\omega E(r,t)$$

Therefore,

$$E = -\frac{1}{i\omega\epsilon_0} \nabla \times \vec{H}$$

To evaluate E, we must determine the curl of *H*, but as in the case of the B-field, only the H_ψ terms have non-zero entries.

From spherical co-ordinates, the only relevant term of the curl of *H* is,

$$(\nabla \times H)_\theta = -\frac{1}{r}\frac{\partial(rH_\psi)}{\partial r}$$

Note also, that the resulting *E*-field is in terms of θ and is perpendicular to the *H*-field, as expected for electromagnetic plane waves.

$$rH_\psi = -i\,\frac{I\Delta l}{2\lambda}\,\sin\theta\left(1-\frac{1}{ikr}\right)e^{-i(\omega t-kr)}$$

$$= -i\,\frac{I\Delta l}{2\lambda}\,\sin\theta\,e^{-i\omega t}\left(e^{ikr}-\frac{e^{ikr}}{ikr}\right)$$

$$\frac{\partial(rH_\psi)}{\partial r} = -i\,\frac{I\Delta l}{2\lambda}\,\sin\theta\,e^{-i\omega t}\frac{\partial}{\partial r}\left(e^{ikr}-\frac{e^{ikr}}{ikr}\right)$$

We solve this using the quotient rule,

$$\left[\frac{u(r)}{v(r)}\right] = \frac{u'(r)v(r)-v'(r)u(r)}{v(r)^2} \qquad u(r) = e^{ikr} \qquad v(r) = ikr$$

$$u'(r) = ik\,e^{ikr} \qquad v'(r) = ik$$

$$\frac{\partial}{\partial r}\left(e^{ikr} - \frac{e^{ikr}}{ikr}\right) = ik\,e^{ikr} - \left(\frac{ik\,e^{ikr}\cdot ikr - ik\cdot e^{ikr}}{(ikr)^2}\right)$$

$$= ik\,e^{ikr}\left(1 - \frac{1}{ikr} + \frac{1}{(ikr)^2}\right)$$

so,

$$\frac{\partial(rH_\psi)}{\partial r} = -i\,\frac{I\Delta l}{2\lambda}\,\sin\theta\,e^{-i\omega t}\,ik\,e^{ikr}\left(1 - \frac{1}{ikr} + \frac{1}{(ikr)^2}\right)$$

and,

$$-\frac{1}{r}\frac{\partial(rH_\psi)}{\partial r} = i^2 k\,\frac{I\Delta l}{2\lambda}\,\frac{\sin\theta}{r}\left(1 - \frac{1}{ikr} + \frac{1}{(ikr)^2}\right)\,e^{-i(\omega t - kr)}$$

we find,

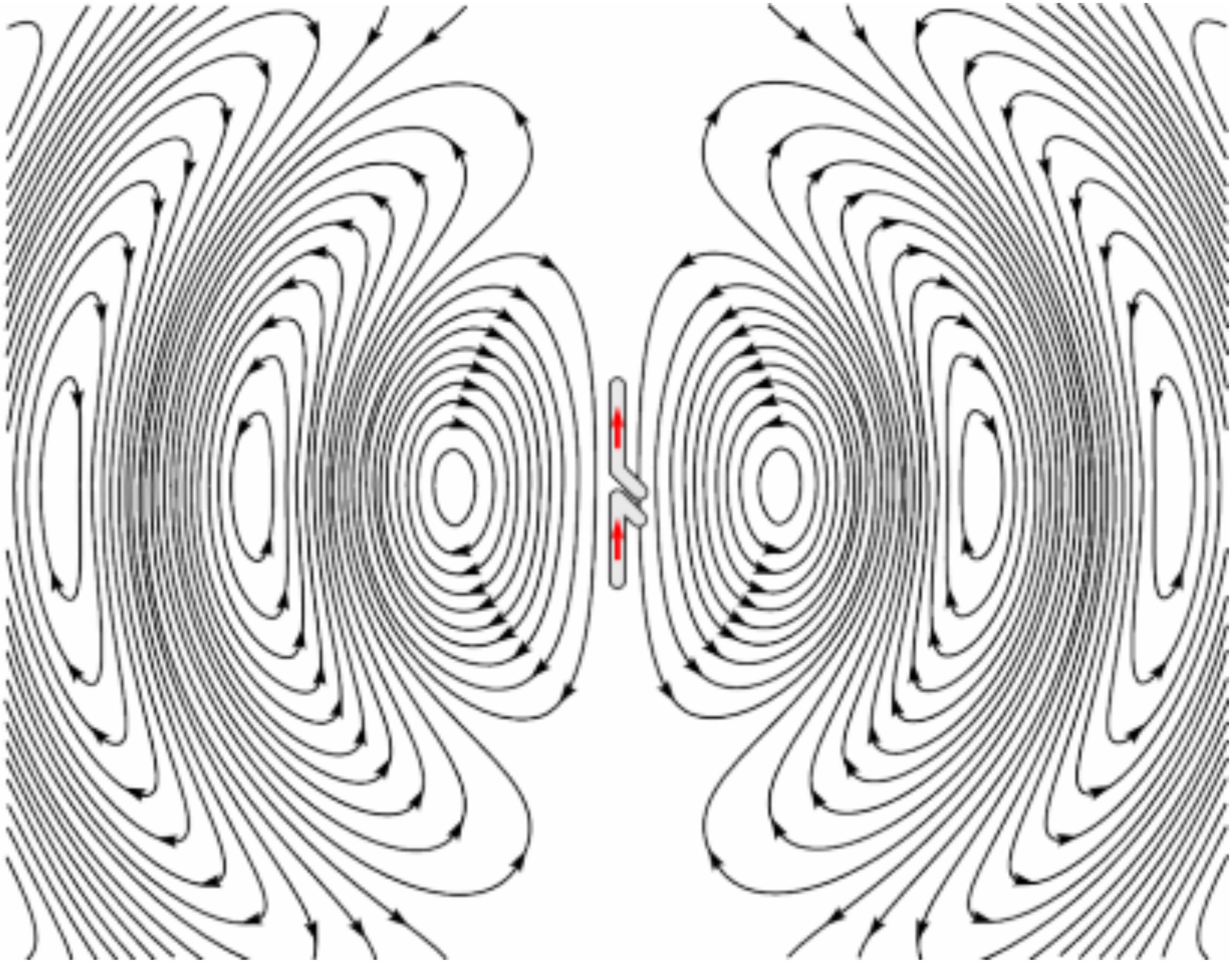
$$k = \frac{\omega}{c}$$

$$E_\theta = -i\frac{1}{c\,\epsilon_0}\,\frac{I\Delta l}{2\lambda}\,\frac{\sin\theta}{r}\left(1 - \frac{1}{ikr} + \frac{1}{(ikr)^2}\right)\,e^{-i(\omega t - kr)}$$

So the E-field can also be expressed as,

$$c = \frac{1}{\sqrt{\mu_0\epsilon_0}}$$

$$E_\theta = -i\sqrt{\frac{\mu_0}{\epsilon_0}}\,\frac{I\Delta l}{2\lambda}\,\frac{\sin\theta}{r}\left(1 - \frac{1}{ikr} + \frac{1}{(ikr)^2}\right)\,e^{-i(\omega t - kr)}$$



So, our electric and magnetic fields are,

$$H_\psi = -i\,\frac{I\Delta l}{2\lambda}\,\frac{\sin\theta}{r}\left(1-\frac{1}{ikr}\right)e^{-i(\omega t-kr)}$$

$$E_\theta = -i\sqrt{\frac{\mu_0}{\epsilon_0}}\frac{I\Delta l}{2\lambda}\,\frac{\sin\theta}{r}\left(1-\frac{1}{ikr}+\frac{1}{(ikr)^2}\right)e^{-i(\omega t-kr)}$$

There are several factors that depend on the power of the distance *r* from the antenna,

1. 1/r : The radiation field (dominates at large $r \gg \Delta l$).
2. $1/r^2$: The induction field
3. $1/r^3$: The static field (of the E-field).

To calculate the near-field properties, all factors must be evaluated, but in the far-field, where we measure the radiation from the antennas, the 1/r term dominates.

$$H_\psi = -i\frac{I\Delta l}{2\lambda}\,\frac{\sin\theta}{r}\,e^{-i(\omega t-kr)}$$

$$E_\theta = -i\sqrt{\frac{\mu_0}{\epsilon_0}}\frac{I\Delta l}{2\lambda}\,\frac{\sin\theta}{r}\,e^{-i(\omega t-kr)}$$

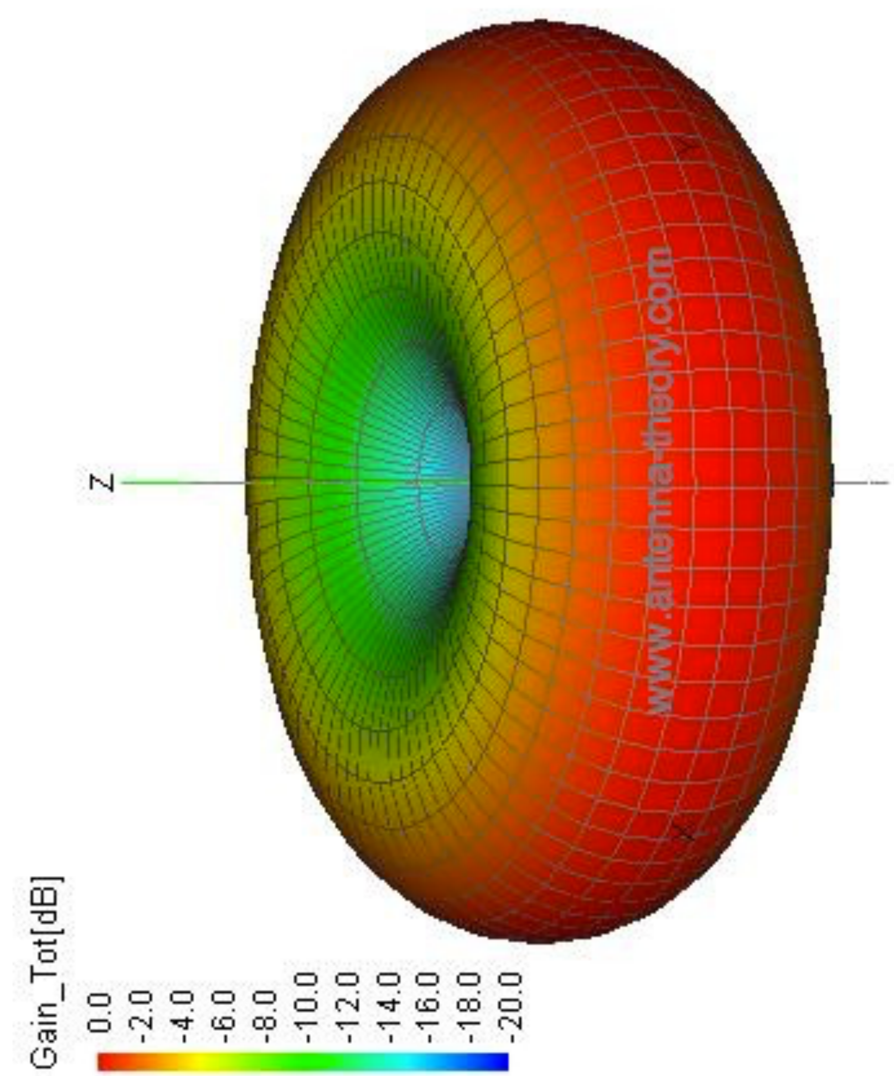
4. We can now determine the directional power per unit area in the far-field by calculating the time-averaged Poynting vector.

$$\langle \vec{S} \rangle = \frac{1}{\mu_0} |\mathrm{Re}\ \vec{E} \times \vec{B}^*| = |\mathrm{Re}\ \vec{E} \times \vec{H}^*|$$

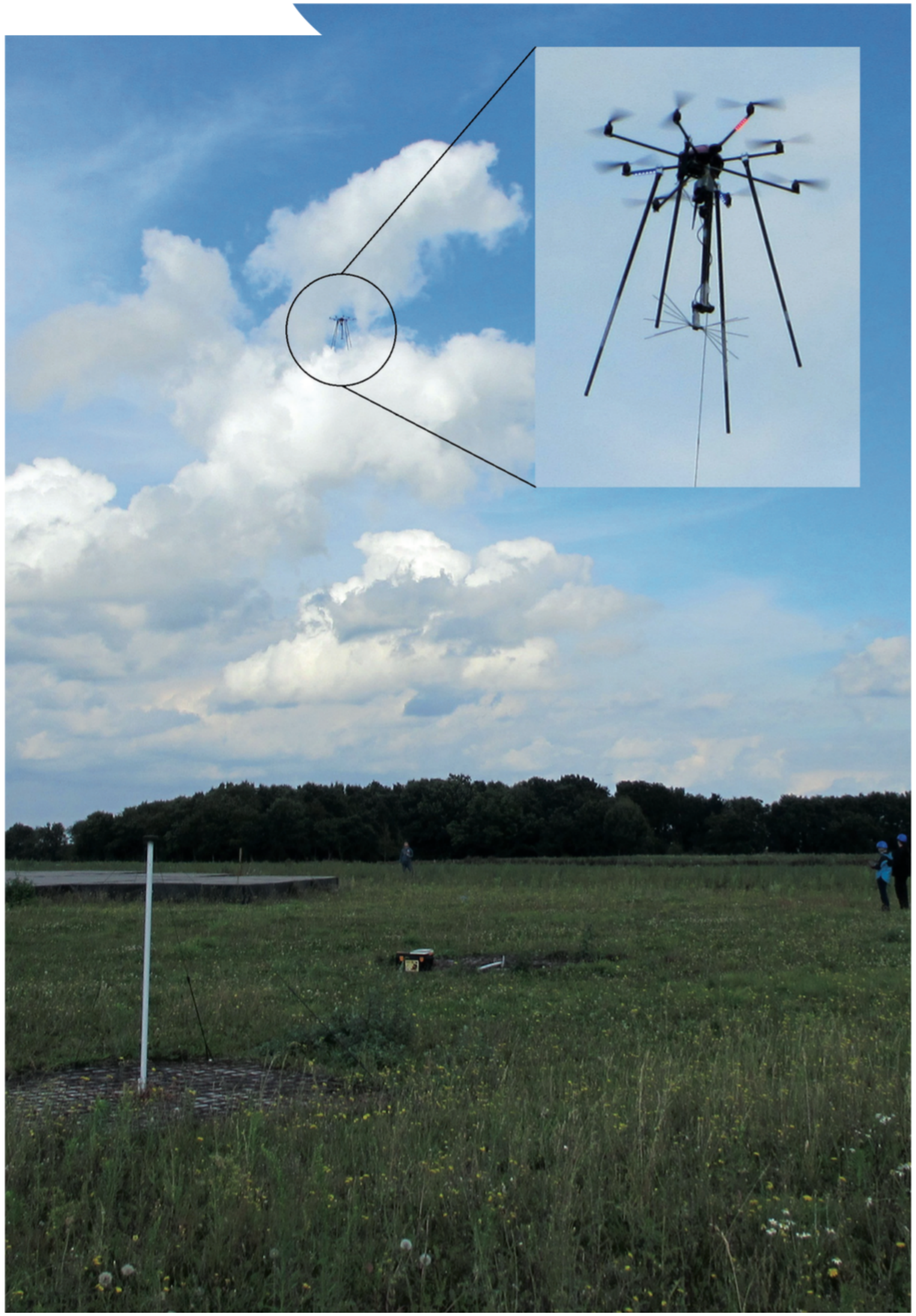
$$= \sqrt{\frac{\mu_0}{\epsilon_0}} \left(\frac{I \Delta l}{2\lambda} \right)^2 \frac{\sin^2 \theta}{r^2} \left(\frac{1}{2} \right)$$

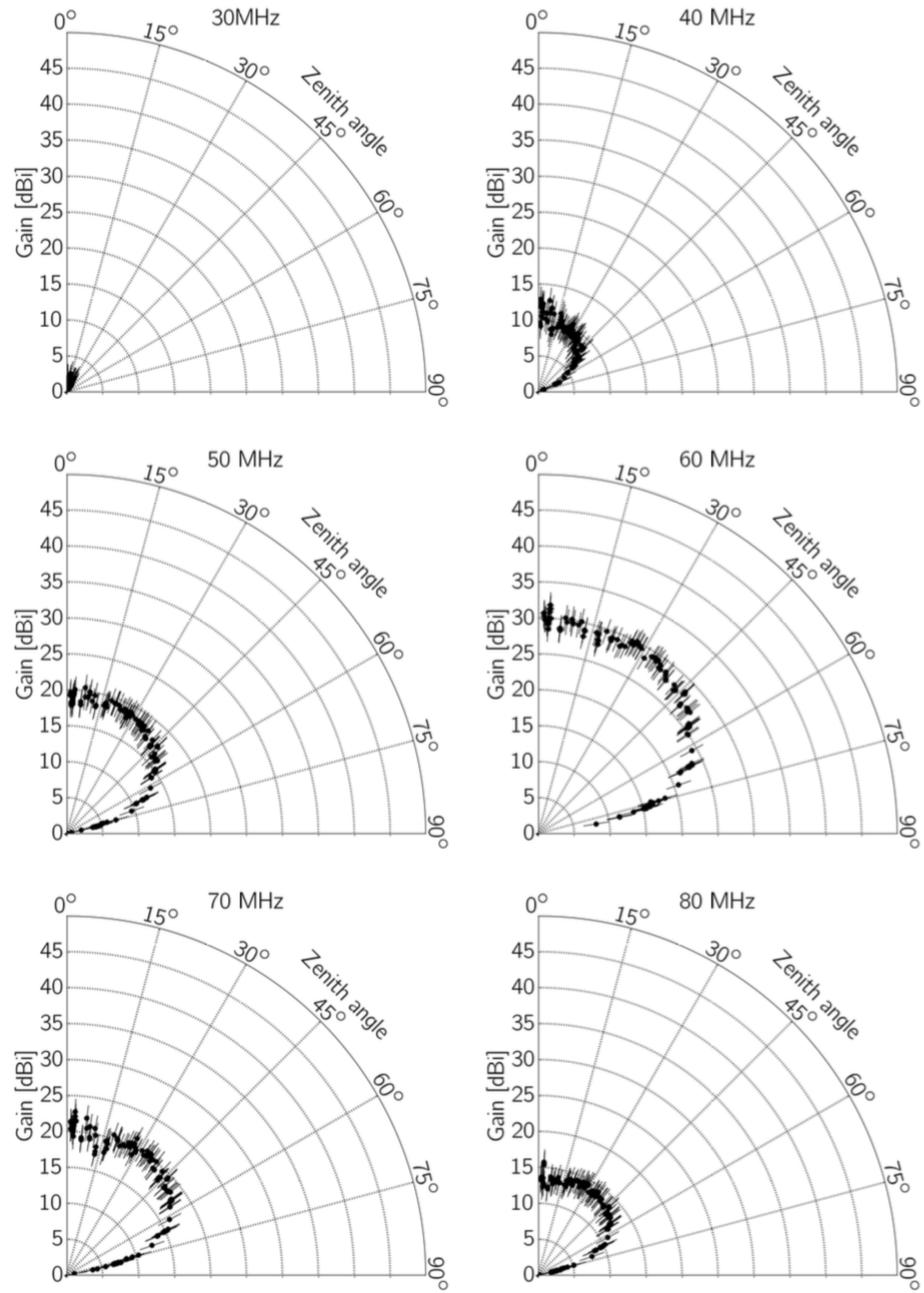
where $\langle \cos^2(\omega t) \rangle = \frac{1}{2}$

The radiation has doughnut shaped power pattern (angular distribution of radiated power) due to dependence on $\sin^2 \theta$.



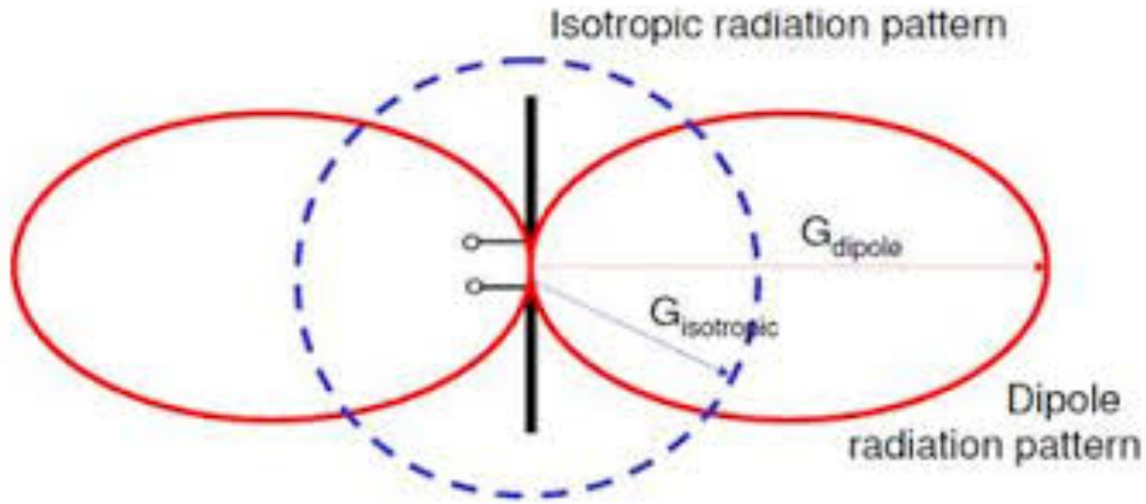
2.2 Response of the LOFAR antenna





2.3 Power gain

G(θ, ϕ) is the power transmitted per unit solid angle in direction *(θ, ϕ)* divided by the power transmitted per unit solid angle from an isotropic antenna with the same total power.



- The power or gain are often expressed in logarithmic units of decibels (dB):

$$G(\text{dB}) \equiv 10 \times \log_{10}(\text{G})$$

Worked example: What is the maximum and half power of a normalised power pattern in decibels?

Maximum power of a normalised power pattern is $P_n = 1$

$$P_n(1) = 10 \times \log_{10}(1) = 0 \text{ dB}$$

Half power of a normalised power pattern is $P_n = 0.5$

$$P_n(0.5) = 10 \times \log_{10}(0.5) = -3 \text{ dB}$$

For a lossless isotropic antenna, conservation of energy requires the directive gain averaged over all directions be,

$$\langle G \rangle \equiv \frac{\int_{\text{sphere}} G d\Omega}{\int_{\text{sphere}} d\Omega} = 1$$

Therefore, for an isotropic lossless antenna,

$$\int_{\text{sphere}} G d\Omega = \int_{\text{sphere}} d\Omega = 4\pi \quad \text{and} \quad G = 1$$

- Lossless antennas may radiate with different directional patterns, but they cannot alter the total amount of power radiated —> the gain of a lossless antenna depends only on the angular distribution of radiation from that antenna.

Key Concept: Higher the gain, the narrower the radiation pattern (directivity).

$$\Delta\Omega \approx \frac{4\pi}{G_{\max}}$$

2.4 Effective collecting area (what is the collecting area of a dipole?)

- The receiving counterpart of the transmitting gain is the effective collecting area, defined as the product of the geometric area and the incident spectral power per unit area (S_ν, the flux-density),

$$A_e \equiv \frac{P_\nu}{S_{(\mathrm{matched})}}$$

Effective area (m^2)

Spectral power (W Hz^{-1})

Flux-density (W m^{-2} Hz^{-1})

Any antenna with a single output measures only one polarisation. Electric fields perpendicular to the antenna wires does not produce currents in the antenna. A pair of crossed dipoles are need to collect the power from both polarisations.

- For an unpolarised source (e.g. like a black body),

$$S_{(\mathrm{matched})} = \frac{S}{2}$$

- The total spectral power from all directions collected by the antenna is,

$$P_\nu = A_e S_{(\mathrm{matched})} = A_e \frac{S}{2} = \int_{4\pi} A_e(\theta,\phi)\frac{B_\nu}{2}d\Omega = kT$$

(must equal the Nyqvist spectral power). From the R-J equation,

$$B_\nu = \frac{2kT}{\lambda^2} \qquad P_\nu = \frac{2kT}{2\lambda^2}\int_{4\pi} A_e(\theta,\phi)d\Omega = kT$$

$$\int_{4\pi} A_e(\theta,\phi)d\Omega = \lambda^2$$

- The average collecting area is defined as

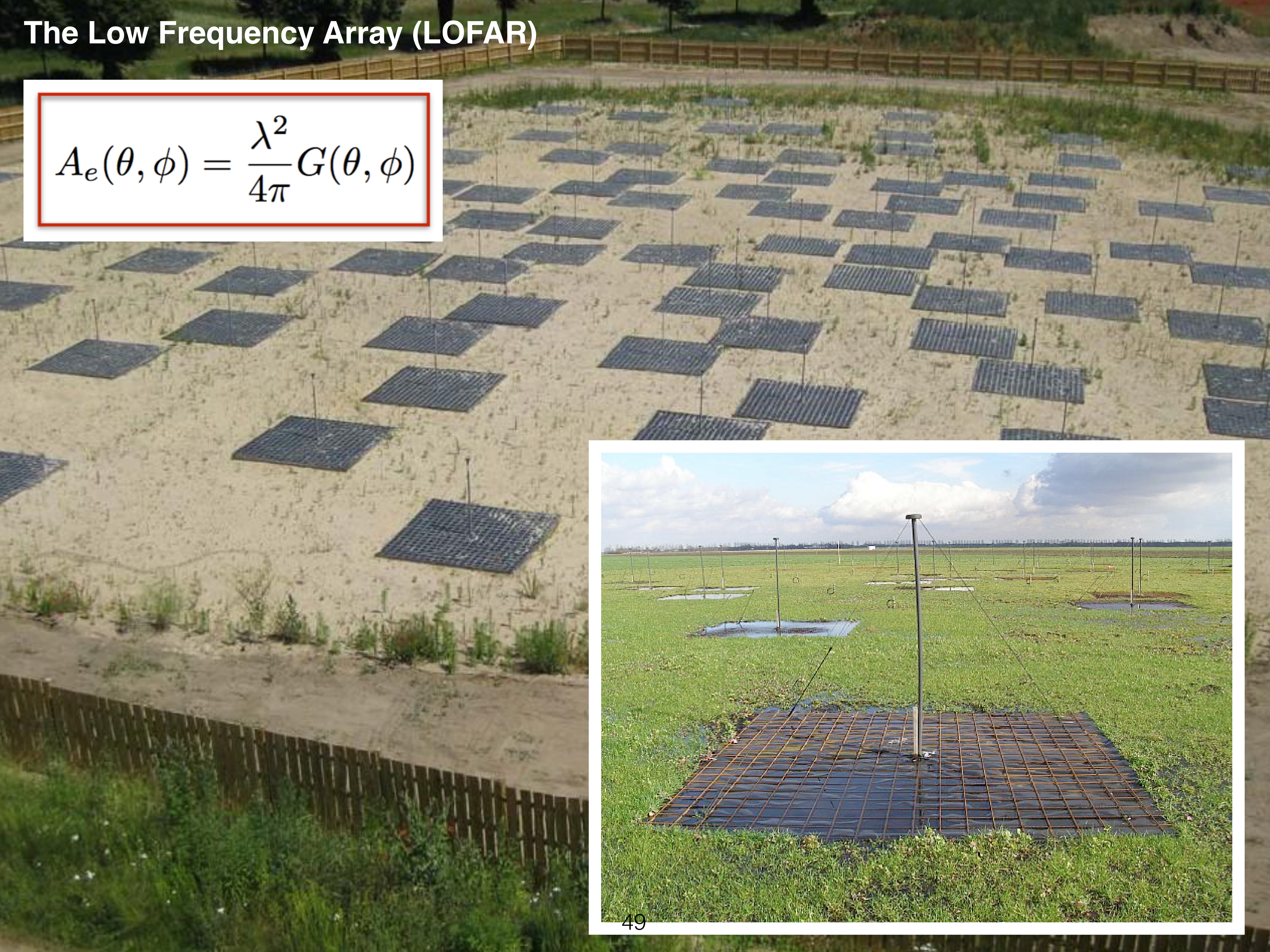
$$\langle A_e \rangle = \frac{\int_{4\pi} A_e(\theta,\phi)d\Omega}{\int_{4\pi} d\Omega}$$

The effective collecting area is independent of the antenna environment, so this relation is valid for any type of radiation (not just black body radiation).

Key concept: Any antenna has the same average collecting area $\langle A_e \rangle$ that depends only on the wavelength of the radiation. $\langle A_e \rangle = \frac{\lambda^2}{4\pi}$

The Low Frequency Array (LOFAR)

$$A_e(\theta, \phi) = \frac{\lambda^2}{4\pi} G(\theta, \phi)$$



The Low Frequency Array (LOFAR)

3. The response of a dish antenna (e-MERLIN, EVN, NOEMA, ALMA)

3.1 Reflector antennas basics

- Paraboloidal reflectors: To be useful at short wavelengths an antenna must have a collecting area > $\lambda^2 / (4\pi)$ of an isotropic antenna and provide a much larger angular resolution (more directive) than a short dipole.

- As arrays of dipole are impractical at λ < 1 m (small effective collecting area), most radio telescopes use large reflectors to collect and focus power onto simple feed antennas (waveguide horns, dipoles) connected to receivers.

Worked example: What is the geometric area of the Dwingeloo telescope at 10 GHz?

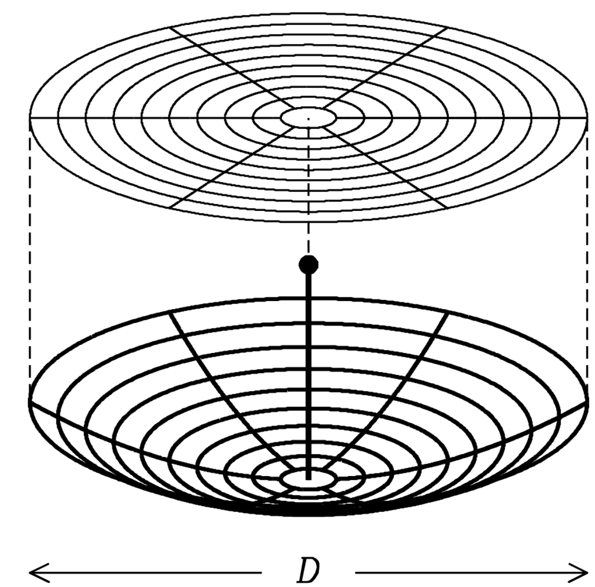
$$A_\mathrm{g} = \pi\left(\frac{D}{2}\right)^2 = \pi\left(\frac{25}{2}\right)^2 = 491\ \mathrm{m}^2$$

This is about 5 x 10^6 times larger than the *effective area* of a short dipole.

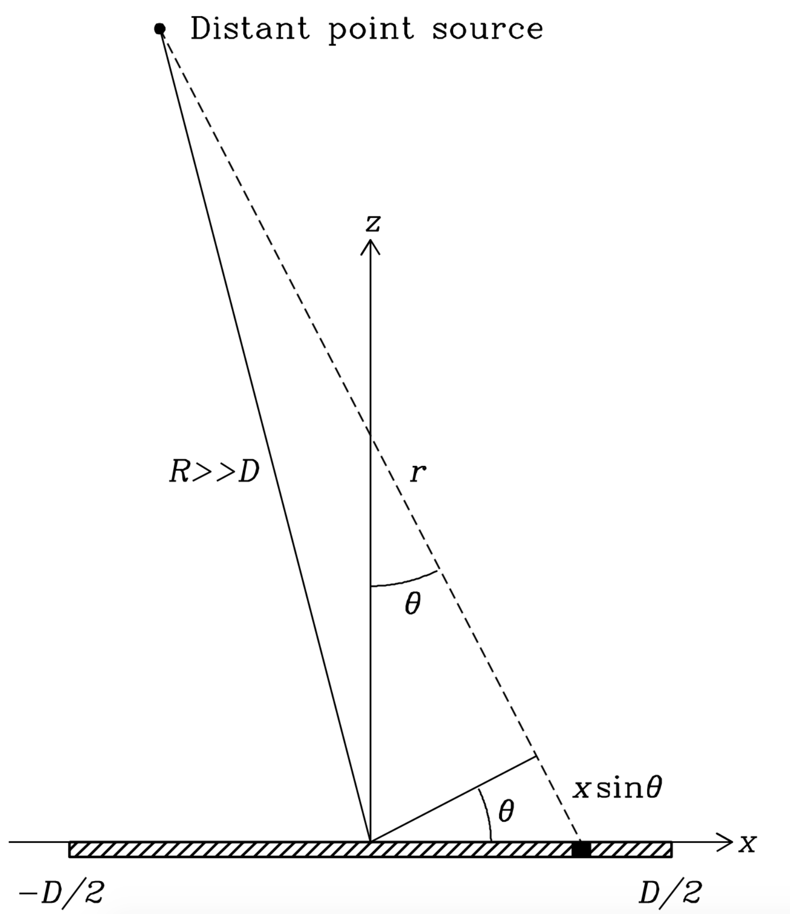
3.2 The aperture illumination pattern

- Aperture pattern: An aperture is the opening which all rays pass. For a paraboloidal reflector of diameter *D*, the aperture is a plane circle with diameter *D*.

- Determining the power gain as a function of position for a circular aperture is complex, so we will make a few simplifying assumptions.



1. Consider a 1-D aperture of width D and calculate the electric field pattern at a distant point ($R \gg R_{ff}$).



Consider a transmitting system with a time varying current density *J*, that also depends on position,

$$J(x,t) = J(x)e^{-i\omega t} \qquad \text{for} \qquad -D/2 \leq x \leq D/2$$

$$J(x,t) = 0 \qquad \text{otherwise}$$

The radiation between two points separated by *x* on the aperture, will travel an extra distance

$$\Delta r = x \sin\theta$$

As with the short dipole, we can determine the exact electric-field, via the vector potential and Maxwell's equations,

$$\vec{A}(x) = \frac{\mu_0}{4\pi}\int\int\int \vec{J}(x)\frac{e^{ik|x-x'|}}{|x-x'|}d^3x' \qquad \vec{E} = -\frac{1}{i\omega\epsilon_0}\nabla\times(\nabla\times\vec{A})$$

The integral over the current density is extremely difficult, except for the simplest cases,

So for simplicity, we use Huygen's principle, that the power measured from our distance source is the sum of each element, such that we can define the current grading as,

$$g(x') = \int_0^N J(x')e^{ik\,x'\sin\theta}\,dx' \qquad \text{(over } N \text{ elements).}$$

Therefore, we can express our electric-field in the far-field as,

$$dE_z(\theta) = K_0\frac{g(x)}{r}e^{-i\omega t}\,dx = K_0\frac{g(x)}{r}e^{-i2\pi r/\lambda}\,dx \qquad \text{(where, } K_0 \text{ is a constant).}$$

At large distances compared with the aperture size ($r \gg D$) we can make the Fraunhofer approximation,

$$r \approx R + x\sin\theta \qquad \text{and} \qquad \frac{1}{r} \approx \frac{1}{R} \qquad \text{(almost constant over the aperture)}$$

Therefore,

$$dE_z(\theta) = K_1\, g(x)\, e^{-i2\pi\, x \sin\theta/\lambda}\, dx \qquad \text{where} \qquad K_1 = K_0 \frac{\exp(-i2\pi R/\lambda)}{R}$$

The electric field over the full aperture is,

$$E_z(\theta) = K_1 \int_{-D/2}^{+D/2} g(x)\, e^{-i2\pi\, x \sin\theta/\lambda}\, dx$$

Lets introduce a change of variable $l = \sin\theta$ and normalised length $u = x/\lambda$.

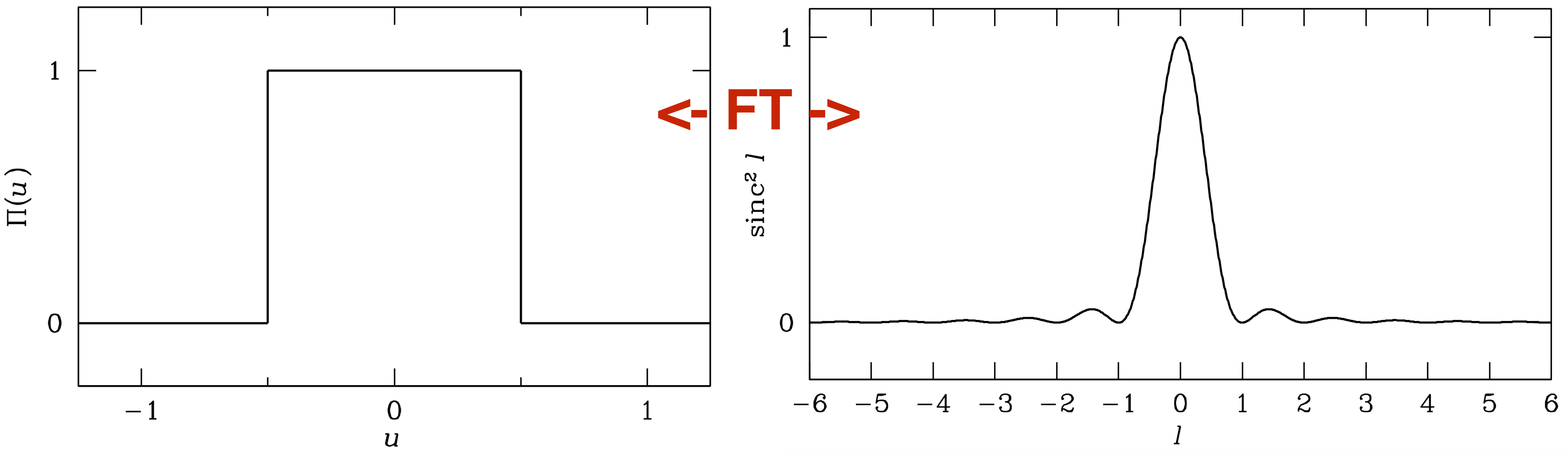
$$E_z(l) = K_1 \lambda \int_{-D/2}^{+D/2} g(u)\, e^{-i2\pi l u}\, du$$

Expressed in a general form (normalised and in units of wavelength) for a receiving antenna, where the electric field pattern is *f(l)* and the electric field illuminating the aperture is *g(u)*,

$$f(l) = \int_{\text{aperture}} g(u) e^{-i2\pi l u} du$$

Key concept: In the far-field, the electric field pattern is the Fourier transform of the electric field illuminating the aperture.

3.3 A one dimensional aperture



- The radiated power as a function of position

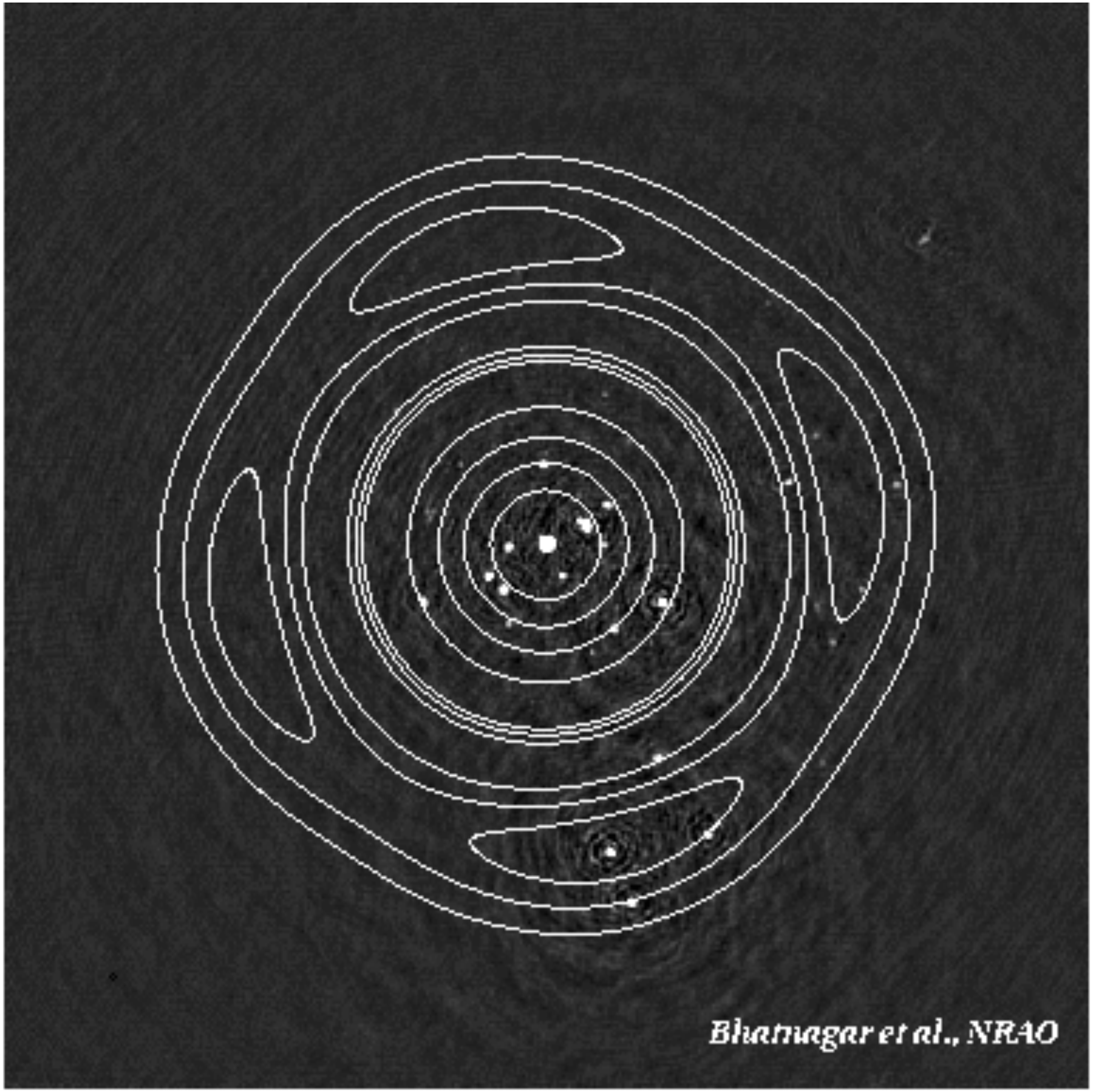
$$P_n(l) = \text{sinc}^2\left(\frac{\theta D}{\lambda}\right)$$

- For a one-dimensional uniformly illuminated aperture,

$$\theta_{\text{HPBW}} \approx 0.89\frac{\lambda}{D}$$

- The central peak of the power pattern between the first minima is called the main beam (typically defined by the half-power angular size).

- The smaller secondary peaks are called sidelobes.


Bhatnagar et al., NRAO

500-metre Aperture Spherical Telescope

Effelsberg Radio telescope: 100 m

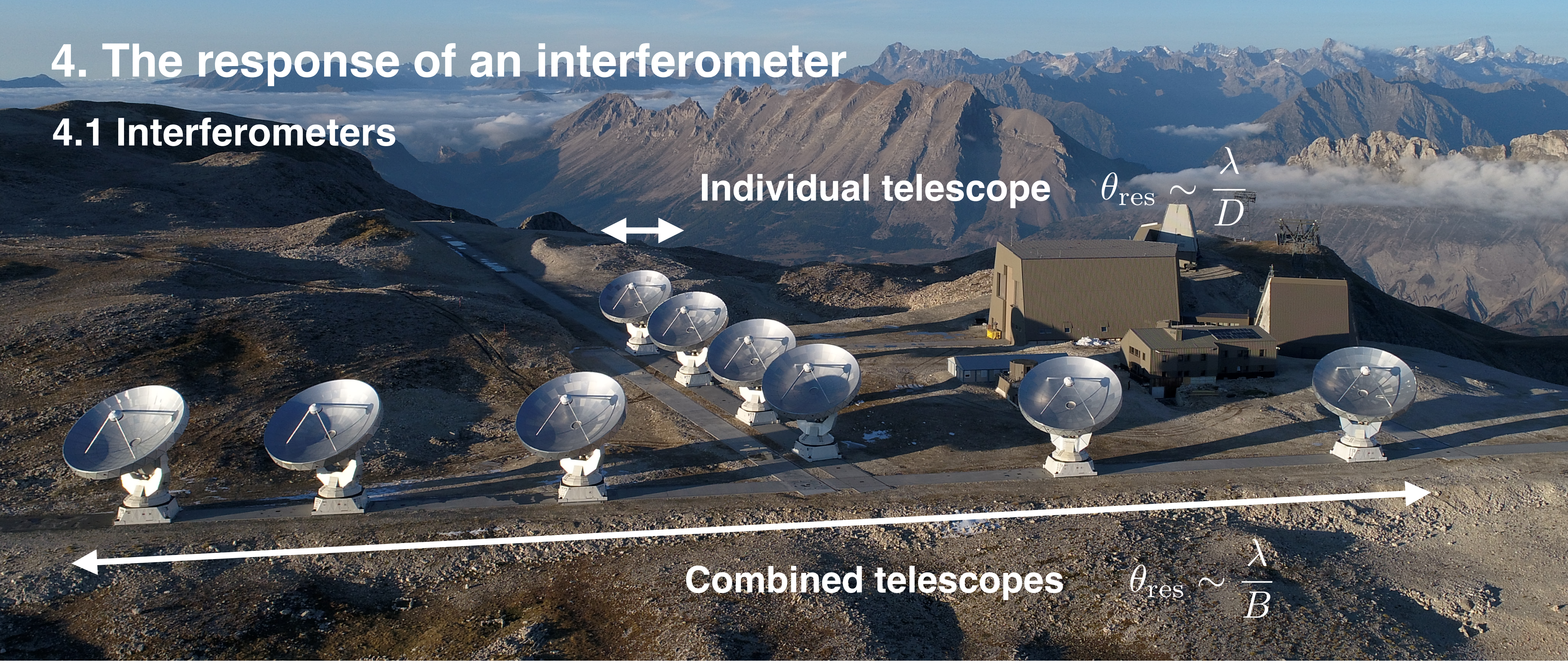
Worked example: What is the spatial resolution (in arcseconds) of the *D* = 500 m FAST radio telescope, operating at *ν* = 5 GHz?

$$\lambda = \frac{c}{\nu} = \frac{3 \times 10^8 \text{ m}}{5 \times 10^9 \text{ Hz}} = 0.06 \text{ m}$$

$$\theta_{\text{resolution}} \sim \frac{0.06 \text{ m}}{500 \text{ m}} \times \frac{180}{\pi} \times 3600 = 25 \text{ arcsec}$$

4. The response of an interferometer

4.1 Interferometers

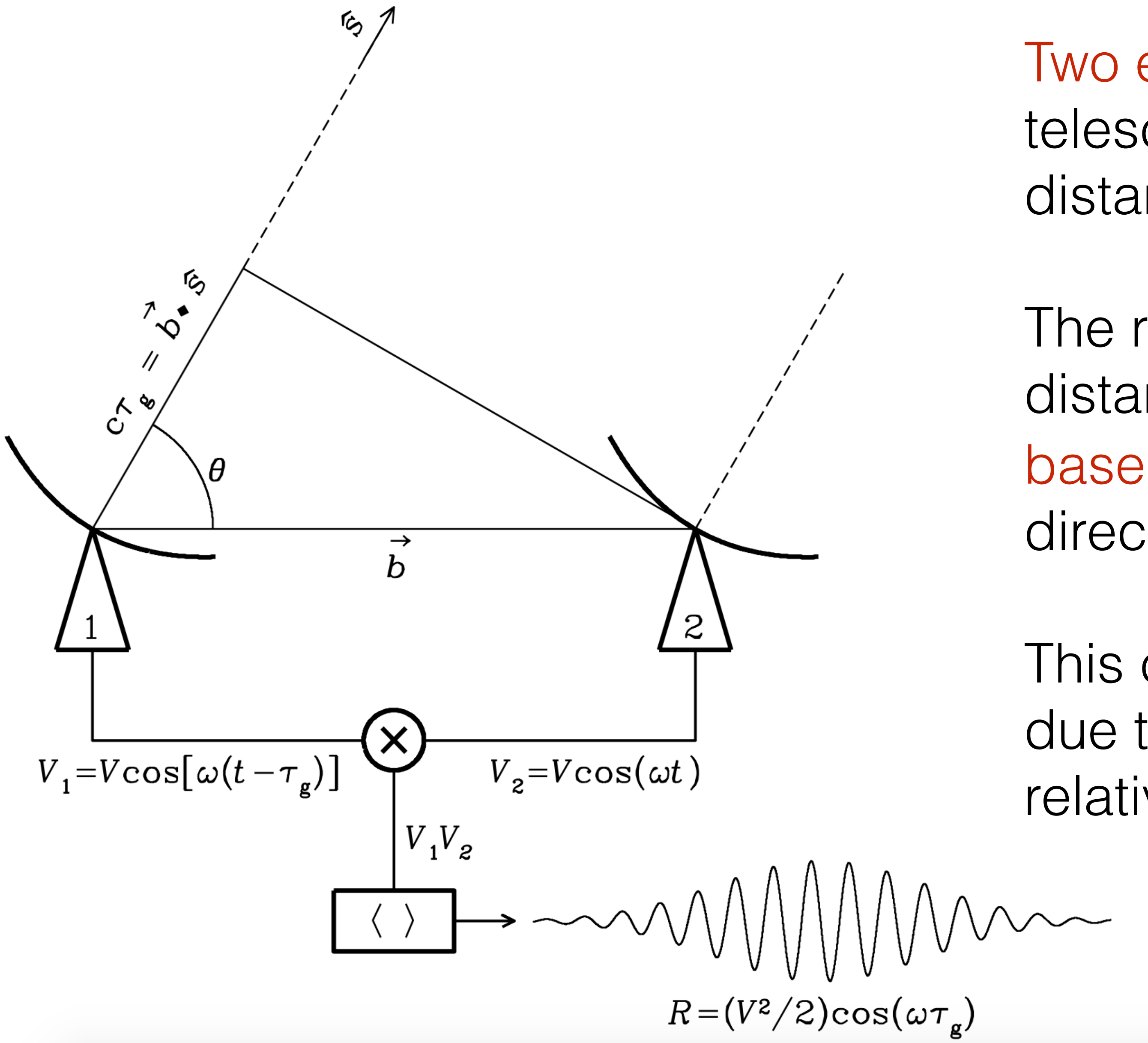


- We can overcome the resolution problem by correlating the signals from different telescopes to effectively increase *D* to an arbitrarily large value by increasing the distance between the telescopes, called the baseline length *B*. Now, $\theta \sim \lambda / B$.

1. High angular resolution (down to < 1 mas; best in astronomy), e.g. EVN, EHT

2. Better sensitivity (Area = $N\pi D^2 / 4$, N is number of telescopes), e.g. LOFAR, e-MERLIN, NOEMA, ALMA.

4.2 A simple two-element interferometer



Two element interferometer: Two identical telescopes observe the electric field of some distant source (c.f. Young's double slit).

The radiation to antenna 1 travels an extra distance $\vec{b} \cdot \hat{s} = b \cos \theta$, where $\vec{b}$ is the vector baseline length and $\hat{s}$ a unit vector in the direction of the source.

This can be expressed as a geometric delay due to the projected position of the source, relative to the baseline of the antennas.

$$\tau_g = \vec{b} \cdot \hat{s}/c$$

For a quasi-monochromatic interferometer (responds to a narrow frequency range ν = 2π / λ), the output voltages over time *t* from the two antennas are,

$$V_1 = V\cos[\omega(t - \tau_g)] \quad \text{and} \quad V_2 = V\cos(\omega t)$$

The correlator multiples the voltages from the two antennas together to give,

$$V_1 V_2 = V^2 \cos[\omega(t - \tau_g)] \cos(\omega t) = \left(\frac{V^2}{2}\right) [\cos[2\omega t - \omega \tau_g] + \cos(\omega \tau_g)]$$

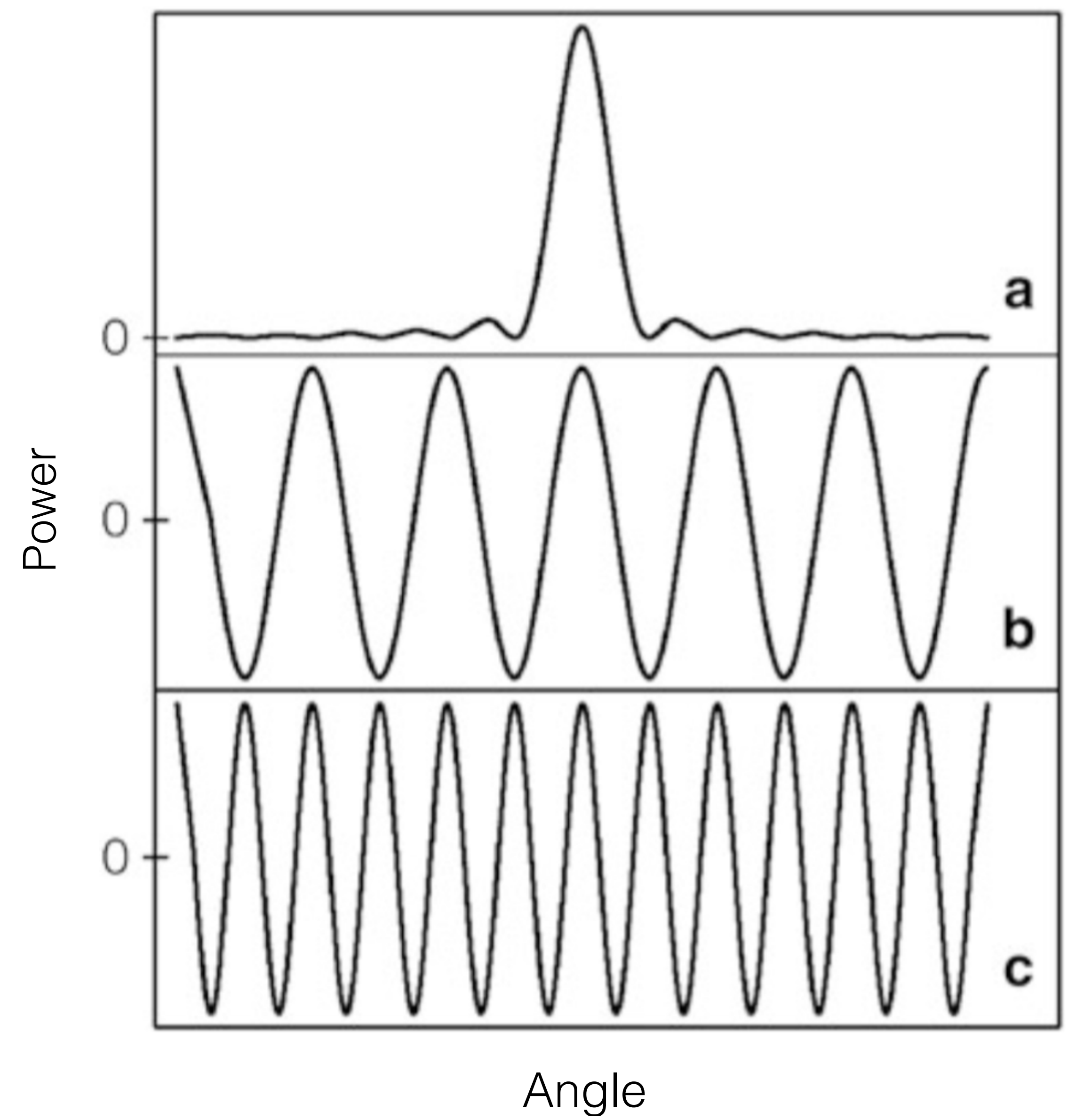
and then a time average [$\Delta t \gg (2\omega)^{-1}$] to remove the high frequency component to give,

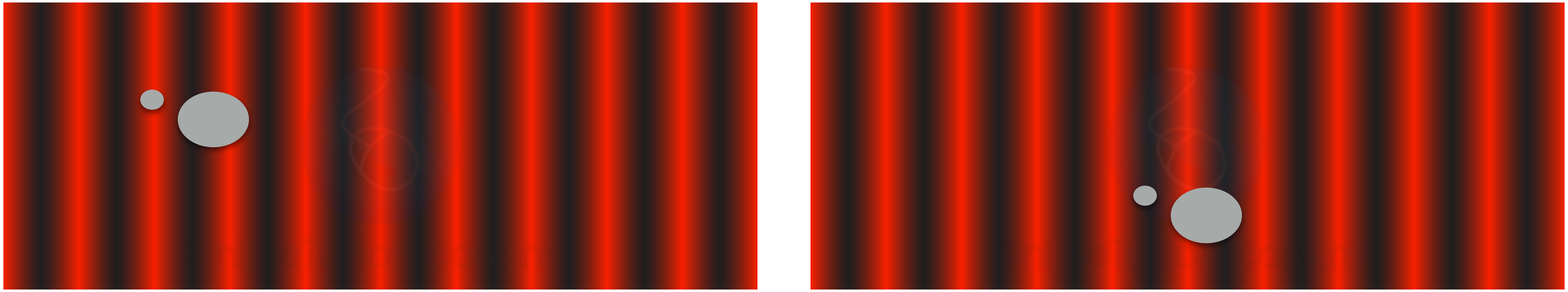
$$R = \langle V_1 V_2 \rangle = \left(\frac{V^2}{2}\right) \cos(\omega \tau_g)$$

a. The power pattern of a filled aperture of diameter *D* with a constant illumination pattern. The FWHM of the main beam is ~ λ / *D.*

b. The power pattern of a two-element interferometer with 2 antennas of diameter *d* and separation *D.* The side-lobe level is constant and the power is centred on 0. The FWHM of the fringes is ~ λ / *D.*

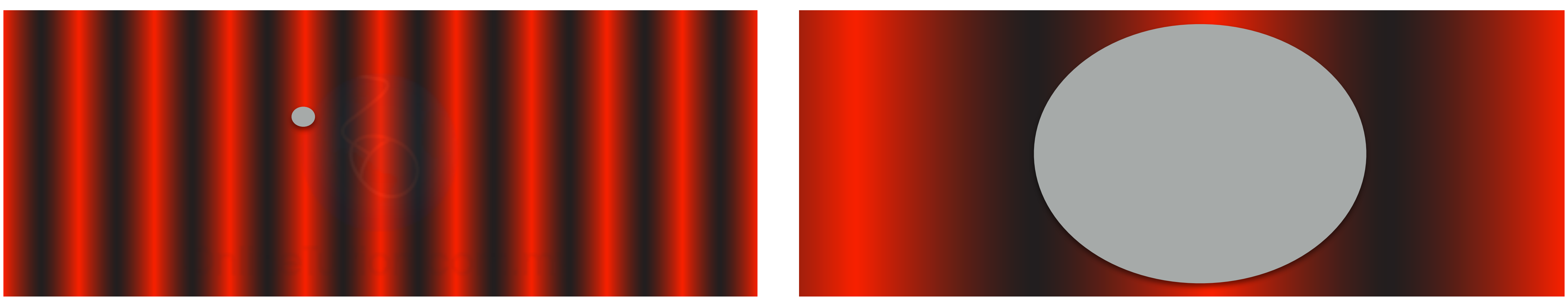
c. The power pattern of a two-element interferometer with 2 antennas of diameter *d* and separation 2*D.* The FWHM of the fringes is now ~ λ / 2*D.*





For sources much larger than adjacent positive and negative fringes, the source cancels itself out and the interferometer response does not vary.

Each set of antennas correspond to a finite set of angular frequencies centred on (b sin θ / λ), small b is needed for extended objects and large b is needed for compact objects.



4.3 Extended sources

A spatially incoherent extended source with sky brightness $I_\nu(\hat{s})$ near frequency $\nu = \omega / 2\pi$ can be considered as the sum of independent point sources. The response of an interferometer is then,

$$R_c = \int I_\nu(\hat{s}) \cos(2\pi\nu\vec{b}\cdot\hat{s}/c)d\Omega = \int I_\nu(\hat{s}) \cos(2\pi\vec{b}\cdot\hat{s}/\lambda)d\Omega$$

Note that, the output from the correlator is a complex quantity and so far we have only considered the (real) cosine part of the signal. The (imaginary) sine component is found by inserting a 90^0 phase delay ($t - \tau_g - \pi/2$).

$$R_s = \int I_\nu(\hat{s}) \sin(2\pi\vec{b}\cdot\hat{s}/\lambda)d\Omega$$

It is convenient to express this in terms of complex exponentials,

$$e^{i\phi} = \cos\phi + i\sin\phi$$

Allowing us to define the complex visibility $V = R_c - iR_s$ as,

$$V = Ae^{-i\phi}$$

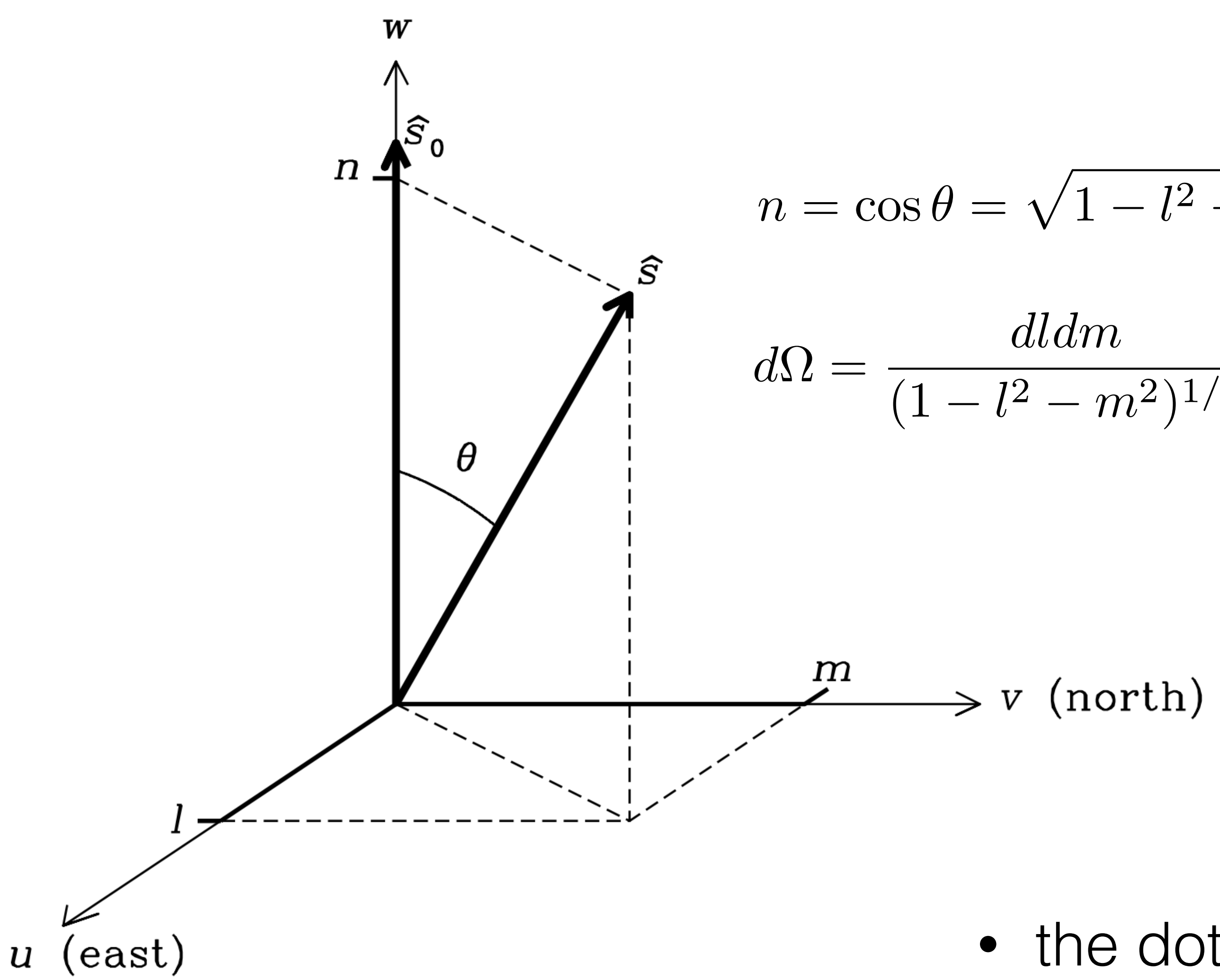
where the amplitude is, $A = (R_c^2 + R_s^2)^{1/2}$ and the phase is, $\phi = \tan^{-1}(R_s/R_c)$

So, we can write the response of a two element interferometer to an extended source with brightness distribution $I_\nu(\hat{s})$ as,

$$V_\nu = \int I_\nu(\hat{s}) \exp(-i2\pi \vec{b} \cdot \hat{s}/\lambda) d\Omega$$

4.4 General response of an interferometer

First, we define our co-ordinate systems.



$$n = \cos\theta = \sqrt{1 - l^2 - m^2}$$

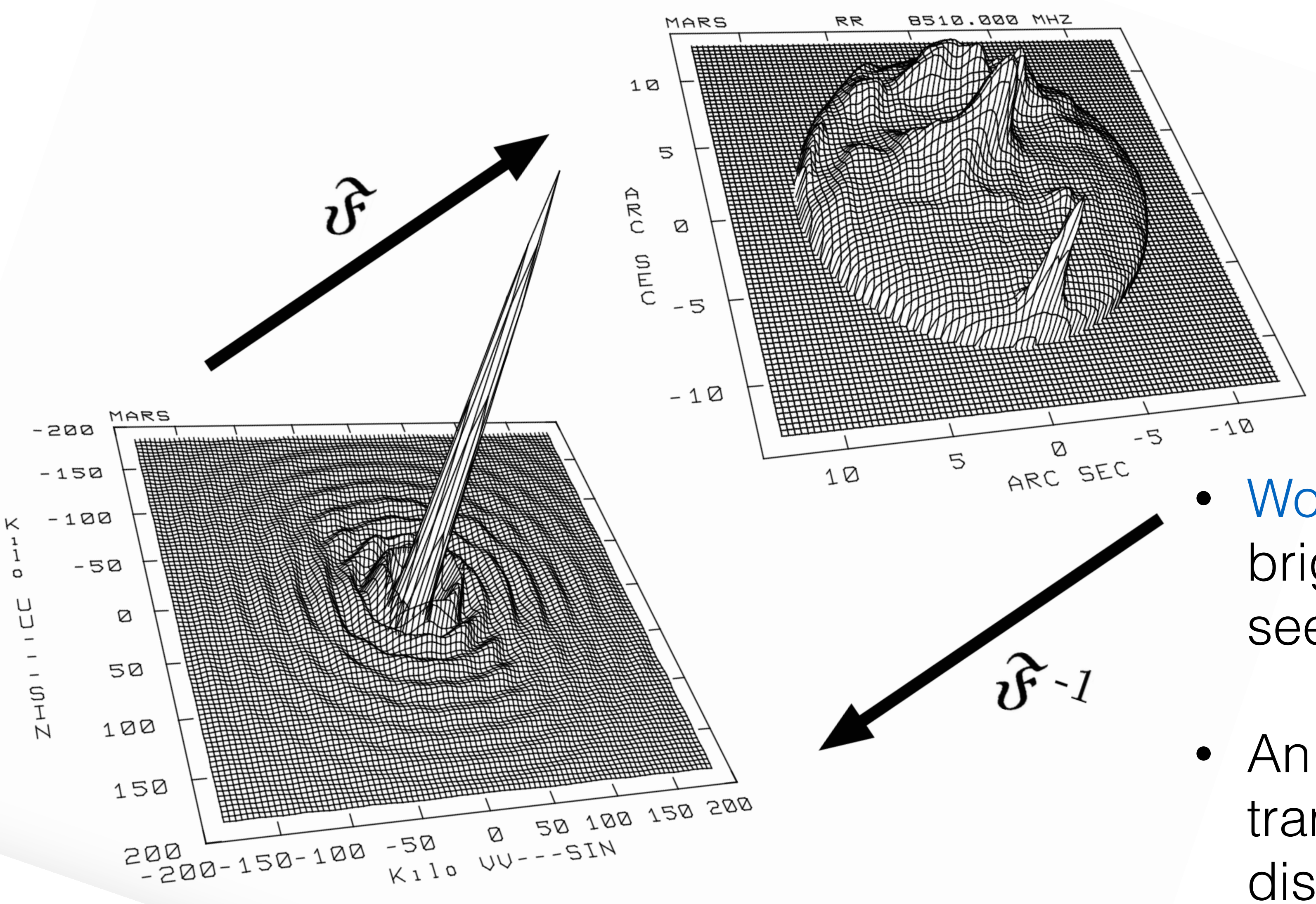
$$d\Omega = \frac{dl dm}{(1 - l^2 - m^2)^{1/2}}$$

- baseline $$\frac{\vec{b}}{\lambda} = (u, v, w)$$
  (u: East-West, v: North-South, w: Up-Down)
- source $$\hat{s} = (l, m, \sqrt{1 - l^2 - m^2})$$
  (l: East-West, m: North-South, $\sqrt{1 - l^2 - m^2}$: Up-Down)
- the dot product $$\frac{\vec{b}}{\lambda} \cdot \hat{s} = ul + vm + w\sqrt{1 - l^2 - m^2}$$

We can then describe the response of an interferometer to any position in the sky as,

$$V_\nu(u,v,w) = \int\int \frac{I_\nu(l,m)}{(1-l^2-m^2)^{1/2}} \exp[-i2\pi(ul+vm+wn)]dldm$$

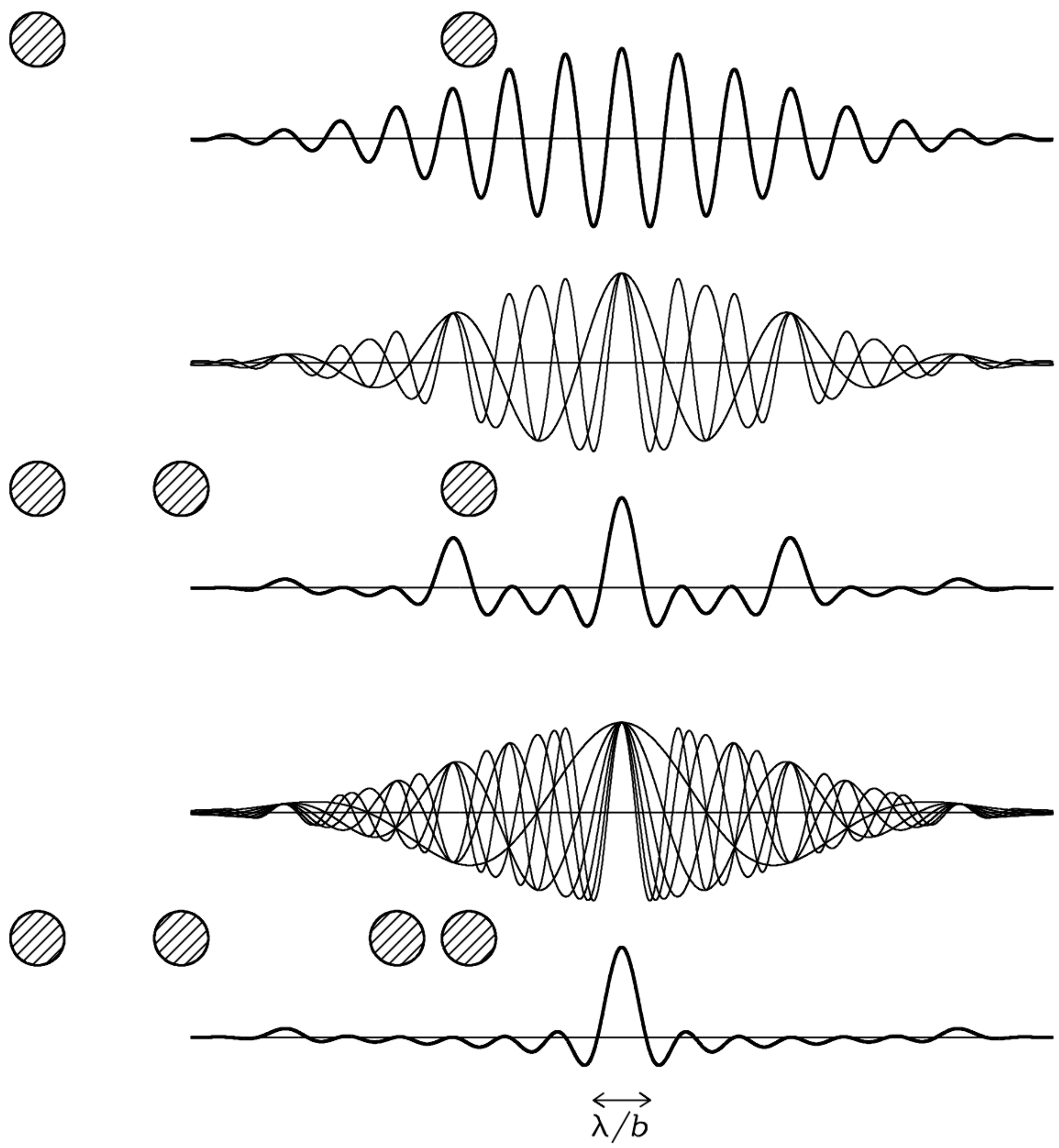
Key Concept: The response of an interferometer is the (inverse) Fourier transform of the (apparent) sky brightness distribution.



- Worked example: Here is the surface brightness distribution of Mars, as seen at 3.6 cm.
- An interferometer will see the Fourier transform of this surface brightness distribution.

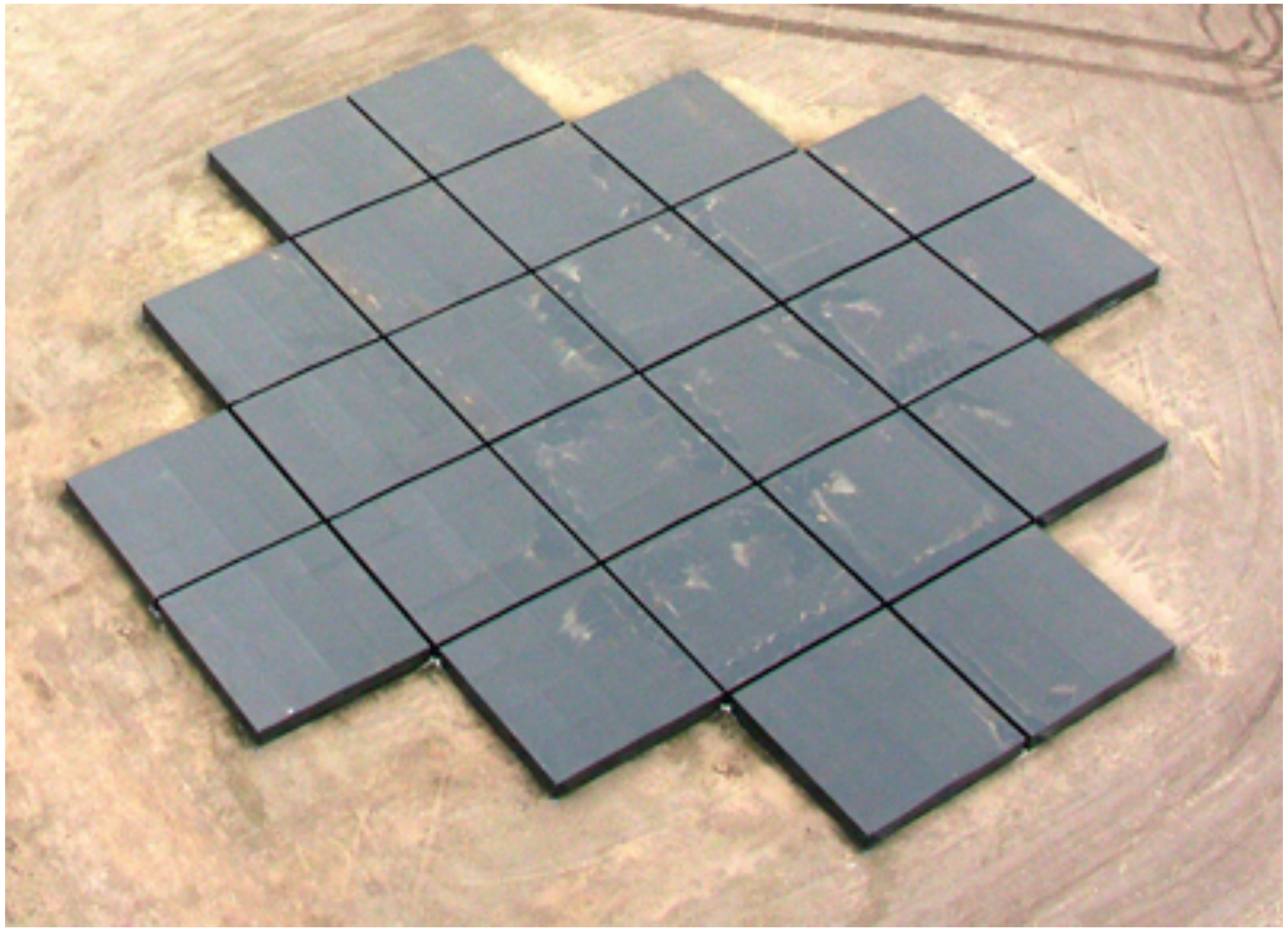
4.5 The synthesised beam (aka. point spread function, dirty beam)

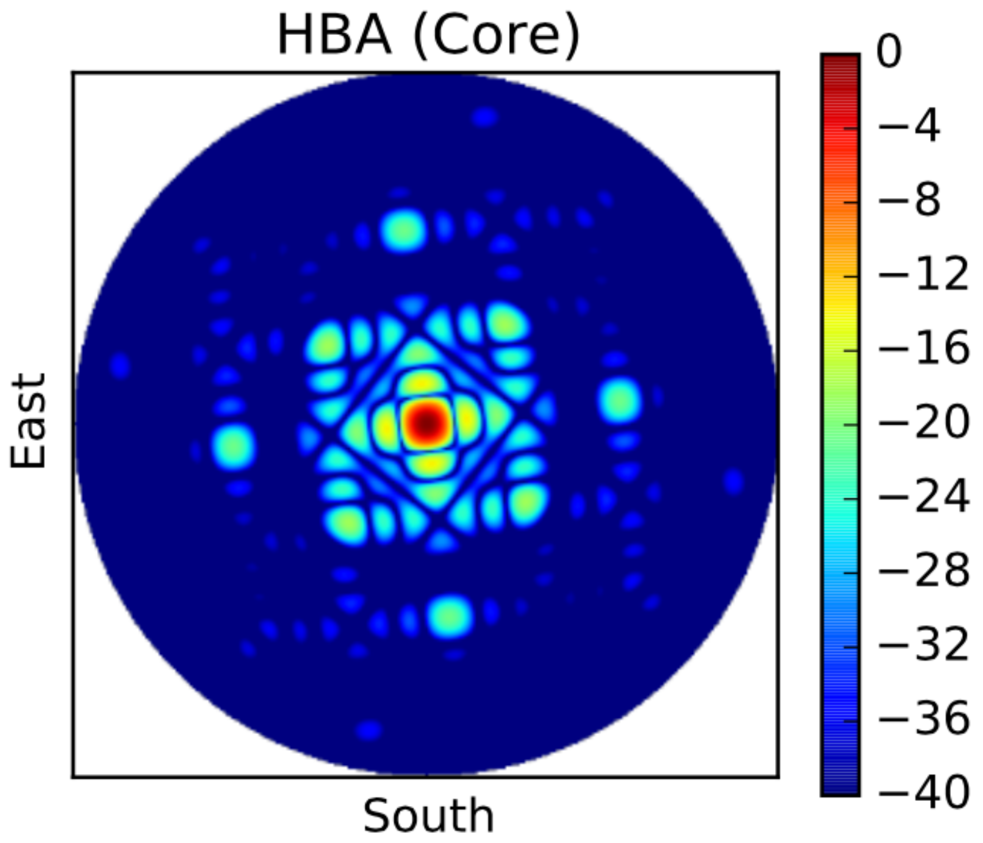
- Incomplete measurement of the Fourier plane results in significant structure in the point spread function response of the interferometer.

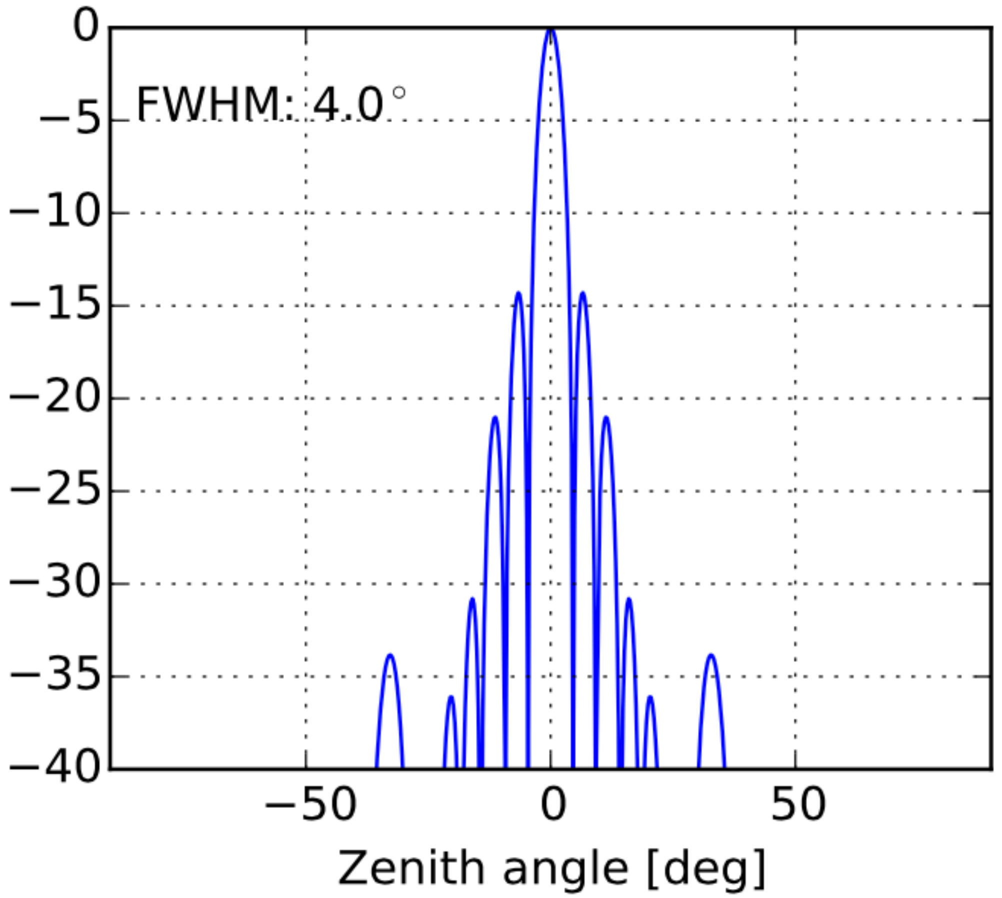


- Two antennas
- Three antennas
- Four antennas

Worked example: The combined 24 tiles of a LOFAR High Band Antenna station (120-250 MHz) arranged in a regular grid.





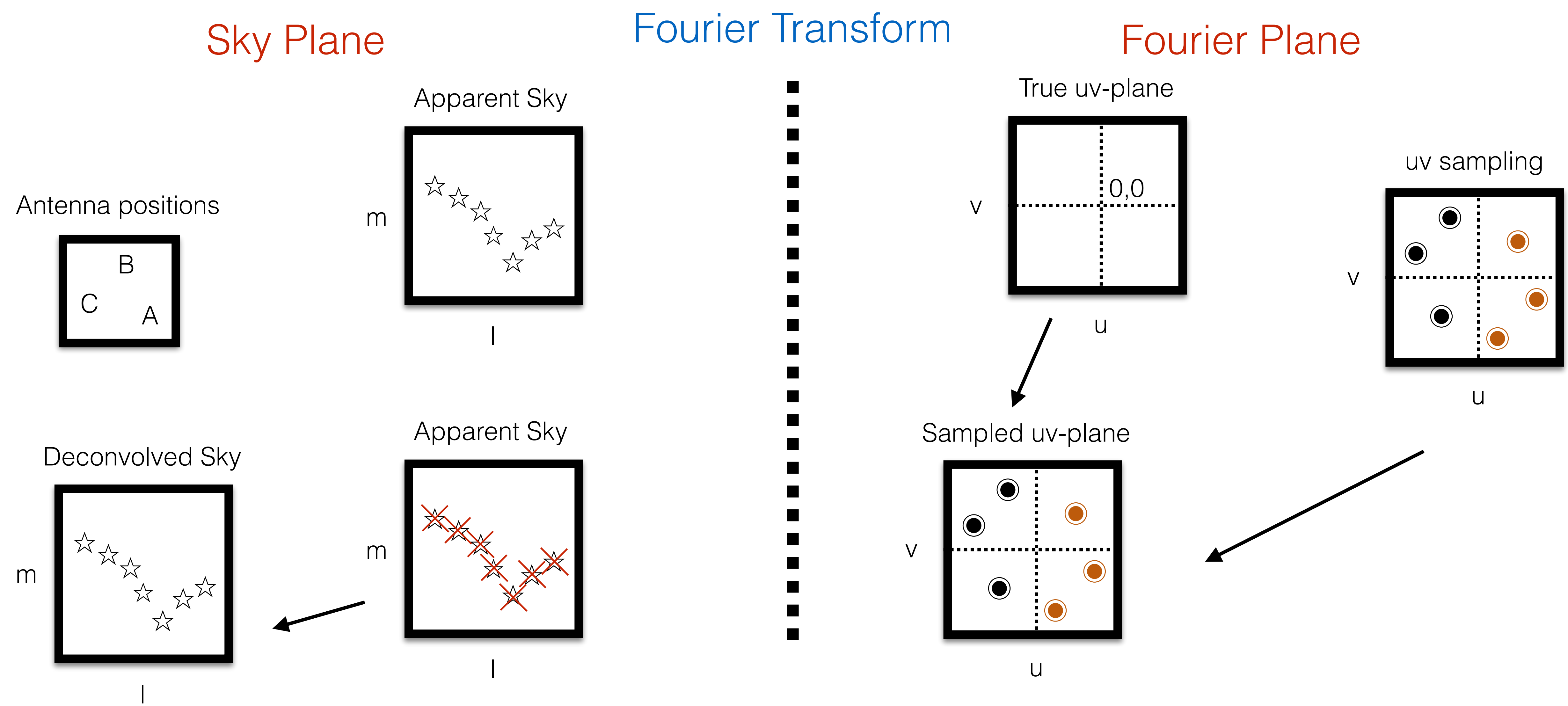


4.6 Visibility Function

Visibility: The data point that each baseline measures is called a visibility (V). Recall, that,

$V = Ae^{-i\phi}$ where the amplitude and phase are, $A = (R_c^2 + R_s^2)^{1/2}$ $\phi = \tan^{-1}(R_s/R_c)$

Each visibility samples a discrete point in the Fourier plane, giving information about the amount of power on some projected angular size on the sky.



Key concept: To make images of the sky that are closest to the true surface brightness distribution we need a completely (well) sampled uv-plane.

4.7 The delay beam

Recall that the simple form of the response of an interferometer to a quasi-monochromatic wave is,

$$V_\nu = \int I_\nu(\hat{s}) \exp(-i2\pi\vec{b}\cdot\hat{s}/\lambda)d\Omega$$

Lets now consider what happens when we increase the integration time and the bandwidth of our observation (which we need to increase the signal to noise ratio and sampling of the *uv*-plane — soon see that σ proportional to $(\Delta\nu * \tau)^{-1/2}$).

For a constant source brightness (doesn't change) over a small bandwidth $\Delta\nu$ centred on frequency ν_c, we can write the response as,

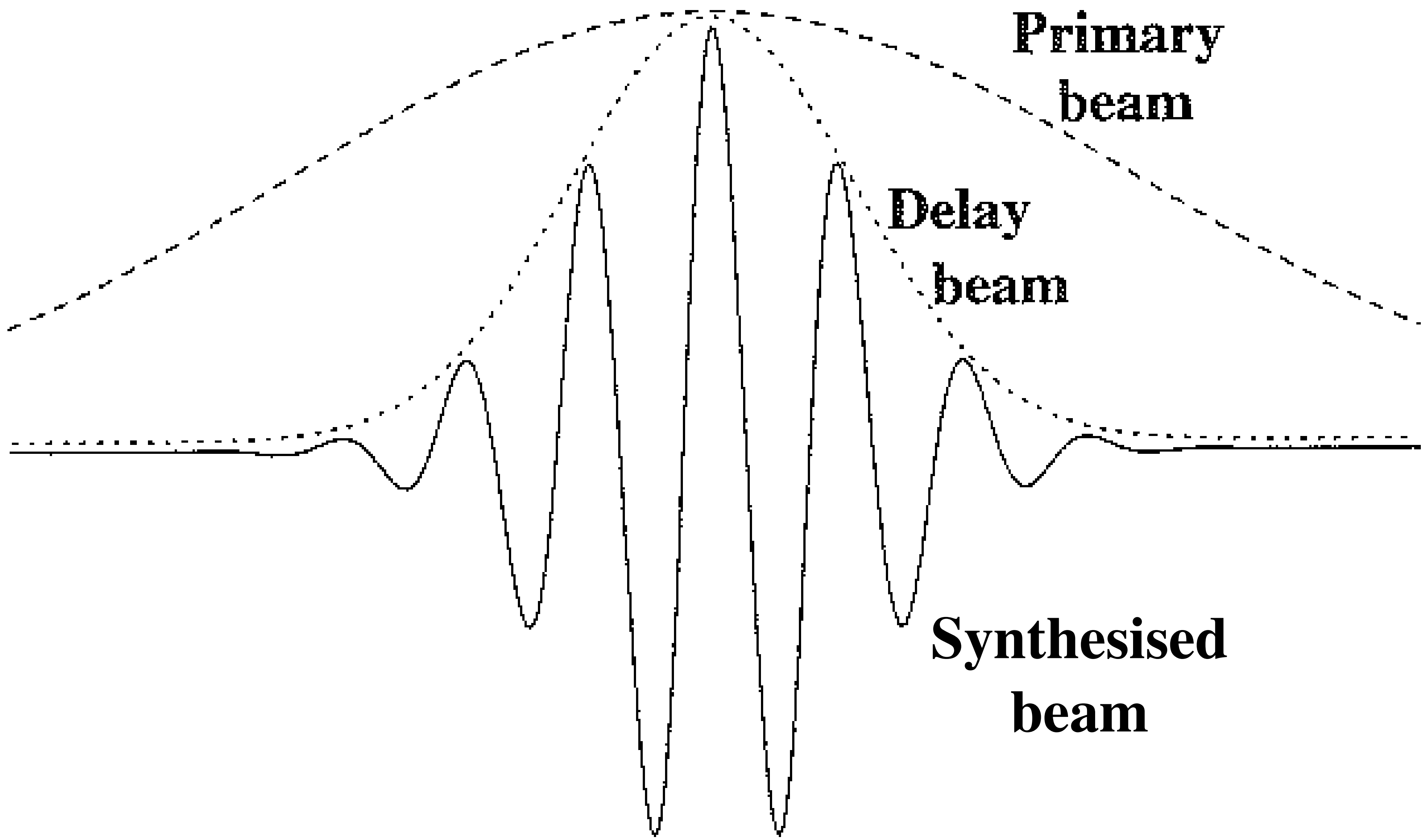
$$V = \int \left[(\Delta\nu)^{-1} \int_{v_c-\Delta\nu/2}^{v_c+\Delta\nu/2} I_\nu(\hat{s}) \exp(-i2\pi\nu\tau_g)d\nu \right] d\Omega \qquad \tau_g = \vec{b}\cdot\hat{s}/c$$

The integral inside the square brackets is just the Fourier transform of a rectangle function,

$$V = \int I_\nu(\hat{s})\text{sinc}(\Delta\nu\tau_g) \exp(-i2\pi\nu_c\tau_g)d\Omega$$

That is, for a finite bandwidth and delay the fringe amplitude is attenuated by a factor $\text{sinc}(\Delta\nu\tau)$ - this is called the delay beam.

5. Summary



We now have three different beams to consider:

1. Primary beam: Due to the power pattern of the individual antennas of the baseline.
2. Synthesised beam: Due to the sinusoidal response of the two elements of the baseline.
3. Delay beam: Due to the attenuation produced by the finite bandwidth of the observation.